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Mathematical introduction to chaos theory

Učební texty k semináři

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Obsah

Kapitola 1

Introduction and History

The aim of this seminar is to give mathematical basic for the study of chaos theory. We will start with a short tour to the history of dynamical systems. We move to simple one-dimensional functions and show how the properties of chaos can appear there. We follow by more complicated two-dimensional case. The next chapter on Chaos gives tools how to study it such as Lyapunov exponents or conjugacy. The sets produced by chaotic behavior, called fractals are studied in the following chapter. The simplest mathematical example is the Cantor set. Maybe surprisingly this set appears naturally in many applications.

The main source for this text is the following book:

K. T. Alligood, T. D. Sauer and J. A. Yorke: Chaos – an introduction to dynamical systems, Springer, 2000.

1.1 A short history tour

Every book on dynamical systems talks about different founders of dynamical systems. We can go back as far as to Sir Isaac Newton, but definitely to Henri Poincaré. Although, the biggest development sure came with computers.

Newton (1643–1727) started describing the physical world with equations. For that, calculus was first needed. Many next generations developped the differential equation studies. But concerning chaotic behavior, it was not yet described in a satisfactory manner.
The physicist James Clerk Maxwell (1831–1879) was probably one of the few people who realized and observed complicated behavior of "chaotic kind".
Henri Poicaré (1854–1912) studied the so called three body problem (simplified solar system with only 3 bodies) and observed the greatly complicated behavior.
Computer pictures brought a great inspiration and new questions.

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One–dimensional maps

A *dynamical system* is a set of possible states (also called phase space or state space), together with a law of evolution (rule) that determines the present state in terms of past states. It means that our process is deterministic.

Mostly, we will use here discrete dynamical systems, where the rule is applied at discrete times (for example n-dimensional maps). The limit of such systems are continuous dynamical systems. Those are differential equations, the most natural way of describing systems since Newton.

2.1 Iteration

As a discrete dynamical system we will use a one-dimensional map (function with domain=range) and its iterations. The n–th iterate of f is the map $f^n = f \circ f^{n-1}$, $n \in \mathbb{N}$. The negative iterates are given by $f^{-n} = (f^n)^{-1}$, $n \in \mathbb{N}$. We use the notation $f^0 = f$. The *(forward) orbit* of a point x is the set $\{f^n(x) \mid n \in \mathbb{N}\}$ where x is called the *initial value.*

We will first show some very simple systems and will look at the orbits of points. The aim is to show that there are very simple orbits but also complicated orbit despite the simplicity of the model.

Example. Population growth.
We can think of the linear map $f(x) = 2x$. Here x denotes the population (in millions). The dynamical system here can be written in the form $x_n = f(x_{n-1}) = 2x_{n-1}$, where n is time and x_n the population at time n.
In this example a nonzero population will grow to infinity. This is called **exponential growth**. Such model is not real for too long. A better model for resource-limited population can be as follows $g(x) = 2x(1-x)$. Compare the models for small and large x! Use a calculator, take the starting point as $x = 0.01$ and discuss the outcome. Notice that the second model gives the same saturation with different starting points.

A point x is called a *fixed point* of a map f if it satisfies $f(x) = x$.

The fixed point of f is 0.

Question. What are the fixed points of g?

Graphical representation of an orbit.
We might use a *cobweb plot* for a rough sketch of an orbit. For our example above it clearly shows that the fixed points are intersections with the diagonal.
We can also make the following simple observation: If the graph is above the diagonal the orbit moves to the right, if below then to the left.
The situation can be much more complicated, we could have more fixed points, more complicated orbit behavior.

2.2 Stability

In this section we will develop the concept of stability that we started to observe in the previous section. Points close to a fixed point can move towards the fixed point or move away from it. Then we can call the fixed point stable or unstable. To illustrate it, we recall that a ball is stable in the valley and unstable on top of the hill. Stability of fixed points is an important property since the real word suffers from small perturbations all the time.

Definition. Let p be a fixed point of a real map. If all points sufficiently close to p are attracted to p, then p is called a *sink* or *attracting fixed point.* If all points sufficiently close to p are repelled from p, then p is called a *source* or a *repelling fixed point.*

In the sequel, unless otherwise stated, we will use *smooth* maps, i.e. continuous with continuous derivatives of all orders.

Theorem. Let f be a smooth map on $\mathbb{R}$ wit a fixed point p.

a) If $|f'(p)| < 1$, then p is a sink.

b) If $|f'(p)| > 1$, then p is a source.

We should check in the logistic example above the type of the fixed points: $x = 1/2$ is a sink and $x = 0$ is a source of g.
Note that for $f(x) = ax$ with $|a| < 1$, the sink at 0 attracts everything!

2.3 Periodic points

We can modify the function g and consider, in general, the logistic family of maps $g_a(x) = ax(1-x)$. The behavior of orbits for different values of a is varied. We can use calculator for experiments (try e.g. $a = 3.3$ where we see two repelling fixed points and most orbits are "attracted"to a couple of points).

Definition. Let f be a real map. We call p a *periodic point of period k* (shortly *period-k point*) if $f^k(p) = p$ and k is smallest such positive integer. The orbit of p is called a *periodic orbit of period k* (shortly *period-k orbit*). The period-k point p is a *periodic sink (source)* if p is a sink (source) for the map f^k.

Stability test for periodic orbits. The k-periodic orbit $\{p_1, \ldots, p_k\}$ is a sink if

$$|f'(p_k) \ldots f'(p_1)| < 1$$

and a source if

$$|f'(p_k) \ldots f'(p_1)| > 1.$$

2.4 The logistic family

Here we want to study in detail the family of maps $g_a(x) = ax(1-x)$ for different values of a. We have already some idea.

When $0 \leq a < 1$, there is a sink at $x = 0$ and every initial point between 0 and 1 is attracted to this sink (i.e. small populations with small reproduction rate die out).

When $1 < a < 3$, there is a sink at $x = (a-1)/a$ (check the derivative!), i.e. small populations grow to a stable state.

When $a > 3$, the fixed point $x = (a-1)/a$ is unstable (check the derivative!), and a period-2 sink appears for $a = 3.3$. When a grows further (approx. 3.45), this period-two sink becomes unstable. For even slightly larger values of a, the situations becomes much more complicated! There are many new periodic points!

When $a > 4$, there are no attracting sets.

Computers can make a bifurcation diagram for $1 < a < 4$. It shows the limiting behavior of orbits (appearance, evolution, disappearance of attracting sets), and also the so called period doubling cascade.

a) Choose a value of a, start with $a = 1$.

b) Choose $x \in [0, 1]$ and calculate its orbit.

c) Ignore the first 100 iterates and plot the iterates starting with 101.

We should further observe and describe the diagram. We need magnifications to study the details (find a period-three sink, for example).

Note 1: The point x was chosen randomly, but nothing changes to the diagram if we pick a different point.

Note 2: For larger a there are many unstable periodic orbits that we do not see in the bifurcation diagram.

Now we will look at the special case with $a = 4$, i.e. $G(x) = 4x(1-x)$. Since this map has no sinks, where do the orbit go? The graph is a simple parabola that everybody knows, but it has rich dynamics. To have an idea, one can first draw graphs of G^2 and G^3. Some simple analysis shows that G^k has 2^k fixed points in the unit interval. It is not difficult to show that the map G has a period-k orbit for any integer k. Hence G has infinitely many periodic orbits.

Question. Is this chaos?

2.5 Sensitive dependence on initial conditions

In this section, we will start with a different kind of map. Let us consider the following map on the unit interval

$$f(x) = 3x \mod 1.$$

This is a discontinuous map but the discontinuity is not an interesting problem here. For our purpose, we could also imagine the map as a map on the circle of circumference 1 and the map is continuous here.

Definition. We call a point *eventually periodic* with period p for the map f if for some positive integer N, $f^{n+p} = f^n(x)$ for all $n \geq N$ and p is the smallest such number.

Note: A point x is eventually periodic for f if and only if it is a rational number.

The map f has the main property of chaos, the sensitive dependance on initial conditions: Taking two points close to each other, they will move apart under iteration.

Definition. Let f be a map on $\mathbb{R}$. A point $x \in \mathbb{R}$ has *sensitive dependance on initial conditions* if there is a nonzero number d (we can call it distance) such that some points arbitrarily close to x are eventually mapped at least a distance d from the corresponding iteration of x. Precisely, there is a number $d > 0$ such that any neighborhood U of x contains a point y such that $|f^k(y) - f^k(x)| \geq d$ for some nonnegative integer k. We call the point x a *sensitive point.*

2.6 Itineraries

To study sensitive dependance on initial conditions, we describe in this section itineraries of orbits. The idea is to use coding (discrete symbols) for orbits that keeps informations.

For the logistic map G, use the symbol **L** to the left subinterval $[0, 1/2]$ and the symbol **R** to the right subinterval $[1/2, 1]$.
For an initial value x we construct a sequence of symbols corresponding to its itinerary as follows: Start with the symbol according to the position of x and continue depending on the position of the following iterates.

We can also use a transition graph to illustrate the possible itineraries. In our case it is a fully connected graph, i.e. all possible sequences of **L** and **R** are possible. Sensitive dependence on initial condition can be easily demonstrated with this concept. For that, we notice that every point in the unit interval has sensitive dependence on initial conditions. We always find a neighbor to x that eventually moves apart by a distance at least $d = 1/4$. Just find the subintervals ...LR and ...RL.

The role of sensitive dependence on initial conditions was understood only with computer simulations.

Use computer to illustrate the sensitive dependence on initial conditions for G!

2.7 Period three

A continuous map that has an orbit of periods three turns to be complicated in the following sense:

1) the map has periodic orbits of all periods, i.e. all possible integers (we state the Sharkovsky theorem later);

2) there is a large set of sensitive points, actually an uncountable set (proved by Li&Yorke in 1975).

The proof of 2) is made using a unimodal map with a period-three orbit and its itineraries as described above. The fact that the infinite set of sensitive points is uncountable is done by a one-to-one correspondence with binary numbers.

Conclusion. Period three implies chaos!!!

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Two–Dimensional Maps

In this chapter we will generalize the concepts from the previous chapter on one-dimensional maps to higher dimensional case. We take the simplest two-dimensional case.

We recall the notion of state first. A state can be a single number (given by a one-dimensional map, single differential equation), more numbers (system of differential equations) but we can also have infinite-dimensional state spaces (given by partial differential equations).

We can have explicit formulas to describe a system (as for example equations describing a falling projectile using Newton's laws of motion). But for real systems, such formulas are rather uncommon. More often we use maps (describing a state in terms of the previous state) or differential equations (formula for the rate of change).

A system of two masses interacting through gravitation can be described by a differential equation. We can find formulas for the orbits of the masses (ellipses). We call such a system analytically solvable. But a system of more masses (we call it n-body problem)is not analytically solvable. We can only approximate and use computers. But it was not known in the past that the exact formulas do not exist.

In 1889 there was a contest concerning the solar system (a special n-body problem). Henri Poicaré was the winner. In his work he made simplifications: He considered two stars moving in circles and an asteroid, all moving in the plane. We call such model planar restricted three-body problem. He observed the orbit of the asteroid (considering the stars in fixed position). Poicaré in his work used the ideas of stable and unstable manifolds (important curves) but he made mistakes. He did not understand well the intersections of these sets, called today homoclinic point and known for they complicated neighborhood (chaos). He realized that before publishing the work and rewrote the paper in 270 pages. The paper contains many important new ideas. One of them is the way he looks at complicated continuous trajectories recording only points of intersection with a plane. This way the dimension of the map is decreased (we obtain a discrete system) but many of its properties are preserved. The new map is called the Poincaré map.

In 1975, M. Hénon presented a two-dimensional quadratic map that displays similar properties as the Poincaré map of differential equations. The map can be given by

$$f(x, y) = (a - x^2 + by, x).$$

It is a nonlinear map. Although the nonlinearity is very "little"it has rich dynamics similar as the logistic map for one-dimensional case. We shall see some computer pictures. Try to make some yourself! And observe that the complexity of some pictures does not decrease with magnification.

Such a two-dimensional map can be also obtained from a differential equation (using T-time map, i.e. values at time intervals of lenght T). Computers are of big use here again. It is possible to use it for the forced damped pendulum.

3.1 Fixed points

Definition. Let f be a map on $\mathbb{R}^m$ and p be a fixed point of f. If there is an $\varepsilon > 0$ such that for all x in the ε-neighborhood $N_\varepsilon(p)$, $\lim_{k\to\infty} f^k(x) = p$, then p is a *sink* or *attracting fixed point*. If there is an ε-neighborhood $N_\varepsilon(p)$, such that each $x \in N_\varepsilon(p) \setminus \{p\}$ eventually maps outside of $N_\varepsilon(p)$, then p is a *source* or *repeller*.

In the two dimensional case we have also a new type of fixed point that cannot appear in dimension one. We call it a *saddle*. It has some attracting direction and some repelling direction.

In the sequel, we will look for ways how to identify sinks, sources and saddles. We will use again derivatives (best linear approximation at the point).

3.2 Linear maps

Definition. A map A from $\mathbb{R}^m$ to $\mathbb{R}^m$ is *linear* if for each $a, b \in \mathbb{R}$ and $u, v \in \mathbb{R}^m$, $A(au + bv) = aA(u) + bA(v)$. It can be represented as a multiplication by an $m \times m$ matrix.

Theorem. Let A be a linear map on $\mathbb{R}^m$ represented by the matrix A. Then

1. The origin is a sink if all eigenvalues of A are smaller than 1 in absolute value;

2. The origin is a source if all eigenvalues of A are larger than 1 in absolute value.

Definition. Let A be a linear map on $\mathbb{R}^m$. We say that A is *hyperbolic* if A has no eigenvalues of absolute value one. If a hyperbolic map A has at least one eigenvalue of absolute value greater than one and at least one eigenvalue of absolute value smaller than one, then the origin is called a *saddle*.

3.3 Nonlinear maps

We will have similar study of stability here as in the one-dimensional case. The stability depends on "linearization"(i.e. derivative).

Definition. Let $f = (f_1, f_2, \ldots, f_m)$ be a map on $\mathbb{R}^m$ and let $p \in \mathbb{R}^m$. The *Jacobian matrix* of f at p is the matrix

$$Df(p) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(p) & \cdots & \frac{\partial f_1}{\partial x_m}(p) \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1}(p) & \cdots & \frac{\partial f_m}{\partial x_m}(p) \end{pmatrix}$$

of partial derivatives evaluated at p.

Theorem. Let f be a map on $\mathbb{R}^m$ and p its fixed point.

1. If the magnitude of each eigenvalues of $Df(p)$ is less than 1, then p is a sink;

2. If the magnitude of each eigenvalues of $Df(p)$ is greater than 1, then p is a source.

Definition. Let f be a map on $\mathbb{R}^m$ and p its fixed point. Then this fixed point is *hyperbolic* if none of the eigenvalues of $Df(p)$ has magnitude one. If p is hyperbolic and at least one eigenvalue of $Df(p)$ has magnitude greater than one and a least one eigenvalue has magnitude less than one, then p is called a *saddle*. (For a periodic point of period k, replace f by f^k.)

Note: Saddles are unstable!

Calculate the Jacobian matrix for the Hénon map with $a = 0, b = 0.4$. Notice that it has two fixed points $(0, 0)$ and $(-0.6, -0.6)$. What is their nature?

To conclude this section, we shall discuss the bifurcation diagram for the Hénon map with $b = 0.4$ and $0 \leq a \leq 1.25$ (horizontal axis). Plot the x-coordinate on the vertical axis.

Note: There are other applications of the Jacobian matrix.

3.4 Stable and unstable manifolds

We know already that a saddle fixed point is unstable. But there are points close to the saddle that do not move away. We will call the points that converge to the saddle the *stable manifold*.

We will not give a precise mathematical definition of a manifold here. An n-dimensional manifold looks locally as the Euclidean space $\mathbb{R}^n$. So, a one-dimensional manifold looks locally like a curve, 2-dimensional manifolds look locally as "disks"or "squares"...

Example. Take the linear map $f(x, y) = (2x, y/2)$. The origin is a saddle with points in the y-axis converging to this saddle (all other diverge to infinity). To see that, we can use the stable manifold. It is the eigenvector of the matrix A corresponding to this map. The eigenvectors are (1,0) corresponding to the stretching by eigenvalue 2 and (0,1) corresponding to the shrinking by eigenvalue $1/2$ (this is the stable manifold of 0). We can also describe the unstable manifold as the stable manifold of the inverse map of f.

Definition. Let f be a smooth one-to-one map on $\mathbb{R}^2$ and let p be a saddle fixed point or a saddle periodic point of f. The *stable manifold* of p, denoted $S(p)$, is the set of points x such that $|f^n(x) - f^n(p)| \to 0$ as $n \to \infty$. The *unstable manifold* of p, denoted $U(p)$, is the set of points x such that $|f^{-n}(x) - f^{-n}(p)| \to 0$ as $n \to \infty$.

When a map is linear, the stable and unstable manifolds of saddles are linear subspaces. For nonlinear maps stable and unstable manifolds cannot be found directly and we need to use computer to approximate them. Recall the Hénon map, it has for some parameter values complicated structure of the stable and unstable manifold, similar as the boundaries of the basins of attracting points.

We notice also a non obvious fact that the stable and unstable manifolds in the plane are one-dimensional sets (lines, curves). And they have a big effect on the dynamics and chaos of the map.

A stable manifold cannot cross itself nor a different stable manifold. But a stable manifold can cross the unstable manifold of the same fixed point (that was, as we discussed already, discovered by Poicaré). The intersection is called a *homoclinic point*. Such intersection necessarily means infinitely many such intersections. We shall also note that the stable and unstable manifolds are invariant sets. We realize here the complexity of the dynamics near the homoclinic points and the existence of sensitive dependence on initial conditions.

Note: Similarly as for the one-dimensional mod-map we discussed in chapter one, we can have a two-dimensional map with infinitely many periods (e.g. defined on a torus).

Question. Is the solar system stable?

Chaos

In many examples (pendulum) unstable behavior is transformed after some time to stable behavior (from source to sink). But we also know already this is not the case always, in some case there is no stable state at all (logistic map).

We can roughly say what is a chaotic orbit:
It keep the unstable behavior (not fixed, not periodic, not eventually attracted to sink).
Arbitrarily close to any its point there are point that move away.

We will introduce in this chapter Lyapunov numbers and Lyapunov exponents to describe this irregularity. We will also state the famous Sharkovskii's theorem.

4.1 Lyapunov exponents

We learned in the previous chapters that stability of fixed and periodic points depends on derivative. If the derivative is $a > 1$ then it means that the orbit of each point close to the fixed point will move away at a rate of approximately a per iteration. For periodic point of period k, we calculate the derivative of the k-th iterate as the product of derivatives at the k points of the orbit (chain rule). If this product is $a > 1$, it means the separation of close point is approximately a per k iterates. So we can think about the average separation rate per iterate. It is easy to see it is $a^{1/k}$ in this case. But we want to have this concept also for points that are not fixed nor periodic.

Definition. Let f be a smooth map on $\mathbb{R}$. The *Lyapunov number* $L(x_1)$ of the orbit $x_1, x_2, \ldots$ is defined as

$$L(x_1) = \lim_{n\to\infty} (|f'(x_1)| \ldots |f'(x_n)|)^{\frac{1}{n}},$$

if the limit exists. The *Lyapunov exponent* $h(x_1)$ is defined as

$$h(x_1) = \lim_{n\to\infty} (1/n)(\ln|f'(x_1)| + \cdots + \ln|f'(x_n)|),$$

if the limit exists.

Note that h exists if and only if L exists and is nonzero, and $\ln L = h$.

Definition. Let f be a smooth map. An orbit $x_1, x_2, \ldots$ is called *asymptotically periodic* if it converges to a periodic orbit. It means that there exists a periodic orbit $y_1, y_2, \ldots y_k$ such that

$$\lim_{n\to\infty} |x_n - y_n| = 0.$$

It is clear that for example an orbit attracted to a sink is asymptotically periodic. The extreme case of the asymptotically periodic is eventually periodic.

Theorem. Let f be a map on $\mathbb{R}$. If the orbit $x_1, x_2, \ldots$ of f satisfies $f'(x_i) \neq 0$ for all i and is asymptotically periodic to a periodic orbit, then the two orbits have the same Lyapunov exponents (if they exist).

4.2 Chaotic orbits

The interesting case is if the orbit is not asymptotically periodic.

Definition. Let f be a map on $\mathbb{R}$. Let $x_1, x_2, \ldots$ be a bounded orbit of f. This orbit is chaotic if

1. it is not asymptotically periodic;

2. the Lyapunov exponent $h(x_1) > 0$.

Before the example, let us repeat the binary expansion of real numbers in the unit interval $[0,1]$. The expansion of a number x is in the form $.a_1a_2\ldots$ with each a_i beeing the 2^{-i} contribution to x. The process goes as follows: multiply the number by 2 and take the integer part as the bit a_1 and repeat the process...this is the map $f(x) = 2x(\mod 1)$!

Simple example. Consider the map $f(x) = 2x(\mod 1)$ on the real line. It has positive Lyapunov exponents and chaotic orbits. Since the map is not continuous, we consider only points that do not map to the point of discontinuity at $1/2$.

$$h(x_i) = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} \ln|f'(x_i)| = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln 2 = \ln 2.$$

Each orbit that does not fall in $1/2$ and is not asymptotically periodic is chaotic!

This map can be best demonstrated by binary expansion of the values. It cuts the left bit:

$$1/5 = .0011\overline{0011}$$

$$f(1/5) = .011\overline{0011}$$

$$f^2(1/5) = .11\overline{0011}$$

$$f^3(1/5) = .1\overline{0011}$$

$$f^4(1/5) = .\overline{0011} = 1/5$$

We easily realize which points are periodic (those with a repeating expansion), eventually periodic (same as asymptotically periodic in this case). We conclude that the chaotic points are the points represented by non eventually periodic binary expansion. Those are the points that are not rational!

Tent map example. $T(x) = 2x$ if $x \leq 1/2$ and $T(x) = 2(1-x)$ if $x \geq 1/2$. We notice immediately the similar shape as the logistic map (but not smooth). We can also use the itineraries of orbits. It is not difficult to realize that the tent map T has infinitely many chaotic orbits (in fact, the absolute value of the slope is 2 except at the peek at $1/2$, therefore the Lyapunov exponent of an orbit is $\ln 2$ if it exists).

4.3 Conjugacy

In the previous section we easily calculated the Lyapunov exponents of the tent map and made immediate conclusion on chaotic orbits. For the logistic map $G(x) = 4x(1-x)$, the slope changes and we cannot make such an easy calculation. But we notice the similarity of both maps (shape, critical point, sources, position of fixed points and orbits, ...). How can we make use of it?

Definition. The maps f and g are *conjugate* if there is a continuous one–to–one change of coordinates, i.e. if $C \circ f = g \circ C$ for a continuous one–to–one map C. This map is called the *conjugacy*.

Check that $C(x) = (1 - \cos \pi x)/2$ is a one–to–one continuous map that conjugates T and G!

Theorem. Let f and g be conjugate maps and C the conjugacy. If x is a period-k point of f, then $C(x)$ is a period-k point of g. If moreover C' is never zero on the periodic orbit of f, then

$$(g^k)'(C(x)) = (f^k)'(x).$$

Consequences:
All periodic points of the logistic map G are sources.
The logistic map G has chaotic orbits.

Note: Unfortunately for most parameter values a in the logistic family no useful conjugacy exists...

4.4 Sharkovskii's Theorem

Sharkovskii's Theorem gives ordering of all possible periods of a map in the sense that if the map has a period-k orbit it also has period-l for any l such that $k \prec l$ (in Sharkovskii's ordering).

the Sharkovskii's ordering is:

$$3 \prec 5 \prec 7 \prec 9 \prec \cdots \prec 2 \cdot 3 \prec 2 \cdot 5 \prec \cdots \prec 2^2 \cdot 3 \prec 2^2 \cdot 5 \prec \cdots$$

$$\cdots \prec 2^3 \cdot 3 \prec 2^3 \cdot 5 \prec \cdots \prec 2^4 \cdot 3 \prec 2^4 \cdot 5 \prec \cdots \prec 2^3 \prec 2^2 \prec 2 \prec 1.$$

Sharkovskii's Theorem. Assume that f is a continuous map on the unit interval and has a period-k orbit. If $p \prec q$, then f has a period-q orbit.

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Fractals

Fractal is a figure that does not simplify when it is magnified. But there is not a unique definition. The term "fractal"was first used in the 1960's by B. Mandelbrot who was employed as a mathematician at IBM. We agree that a fractal has some of the following properties:
Complicated structure (while scaling), repetition of structures (self-similarity), fractal dimension not an integer.

5.1 Cantor set

Cantor set is a theoretical geometrical construction and the simplest fractal. We will describe the construction of the *middle-third Cantor set.*

We start with the unit interval $[0, 1]$ and we will follow the instructions:

Remove the open interval $(1/3, 2/3)$ i.e. the middle-third of I. We call the remaining set $C_1 = [0, 1/3] \cup [2/3, 1]$.

Remove the middle-thirds of the remaining intervals in C_1, i.e. $(1/9, 2/9) \cup (7/9, 8/9)$. The remaining set $C_2 = [0, 1/9] \cup [2/9, 3/9] \cup [6/9, 7/9] \cup [8/9, 1]$.

Repeat this process of removing the middle-third open intervals. The limiting set of this precess is called the *middle-third Cantor set* and is denoted by $C = C_\infty$.

We can now ask many question about the resulting set C. What is its length? Size? Measure? Etc.

The set C is a subset of C_n for each n. We can see by simple induction that the set C_n consists of 2^n intervals each of length $(1/3)^n$ (i.e. of total length $(2/3)^n$). This means that C can be covered by intervals C_n of total length as small as we wish. Therefore, $C = C_\infty$ has length zero! In the precise language of measure theory, C has measure zero.

But C contains many points! And moreover, the endpoints of the intervals are only a small portion of the points in C. For example the point $1/4$ also belongs to C. To realize this, we shall use the ternary representation of points in C (the process of finding it is the same as for binary representation using digits $0, 1, 2$ in this case). We put $c = .c_1c_2c_3\ldots$. It is straightforward to realize that the set C_1 consists of all numbers in the unit interval that can be expressed with $c_1 = 0$ or 2. Same argument works for C_2, i.e. the first and second digit is 0 or 2. And so on! We obtained the next theorem:

Theorem. The middle-third Cantor set C consists of all numbers in the unit interval that can be expressed in base 3 using only the digits 0 and 2.

Note: the number $.\overline{02} = 1/4$ is in C.

The previous theorem shows more. Replacing the digit 2 in each ternary representation by the digit 1 we obtain all possible numbers in $[0, 1]$. That means there is a one-to-one correspondence between the unit interval and C. Hence, C is an uncountable set! The idea of countable and uncountable sets comes also from Cantor.

We also observe that the Cantor set is self-similar (magnifying a part of the set gives again the entire Cantor set). We can also observe this property of fractals looking at the Hénon map and its attractors or boundaries between basins for example (for some parameters).

There are also other methods that produce the Cantor set. For example we can have probabilistic processes (as attractors).

There are many other Cantor sets. In dimension two, the most famous are the Sierpinski gasket and and the Sierpinski carpet.

5.2 Fractals produced by maps

We return to the tent map in a general form. For $a > 0$, we define the tent map with slope a by

$T_a(x) = ax$ if $x \leq 1/2$ and $T_a(x) = a(1 - x)$ if $x \geq 1/2$.

We can describe its behavior for different values of a. Let us consider the case $a = 3$. It has different dynamics than the slope-2 map that is conjugate to the logistic map.

First, we will look at the basin of infinity (points that diverge to $-\infty$). Directly from the graph it is clear that the intervals $(-\infty, 0)$ and $(1, +\infty)$ converge to $-\infty$. It is true also for the interval $(1/3, 2/3)$ since it is mapped to $(1, +\infty)$. If we look further what is mapped to the previous intervals, we find the intervals $(1/9, 2/9)$ and also $(7/9, 8/9)$, and so on. This way we can see that the basin of infinity is exactly the middle-third Cantor set.

The points that are in C travel around C. It can be well described using a 3-base representation.

We finish this section by a short introduction to other fractal sets concerning the basin boundaries.

Mandelbrot set. We consider a complex quadratic map $P_c(z) = z^2 + c$. The orbit of 0 has an important role here (for some parameter c is is a fixed point, for some not...). We define the *Mandelbrot set* by

$$M = \{c : 0 \text{ is not in the basin of infinity for the map } P_c\}.$$

Julia sets. The boundary of the basin of infinity is called the *Julia set*. It is defined equivalently (for polynomials) as the set of repelling fixed and periodic points together with the limit points of this set.

Use computer to draw some Julia sets!

5.3 Fractals dimension

We want to measure the fractal sets using a grid. We count the grid boxes covering the set, then we observe how the number varies changing the grid size.

For an interval J the number of boxes of size ε needed to cover it is no more than $L(1/\varepsilon)$, where L is a constant depending on the length of the interval. Similarly in dimension d, the number is $L(1/\varepsilon)^d$. Now we can take different objects and ask how many boxes we need to cover them. If we denote by $N(\varepsilon)$ the number of boxes of side-length ε needed to cover a given set S, we want to say that S is d-dimensional when

$$N(\varepsilon) = L(1/\varepsilon)^d.$$

We notice, that d can be other than integer. Solving for d we obtain:

$$d = \frac{\ln N(\varepsilon) - \ln L}{\ln 1/\varepsilon}.$$

If L is a constant, we can omit it in the previous formula for small ε.

Definition. A bounded set S in $\mathbb{R}^m$ has *box-counting dimension*

$$\operatorname{boxdim}(S) = \lim_{\varepsilon \to 0} \frac{\ln N(\varepsilon)}{\ln 1/\varepsilon},$$

when the limit exists.

Check that the definition gives correct results for a line segment in the plane of length l. Depending on the position of the line in we plane we need at least l/ε (lies horizontally or vertically) and at most $2l/\varepsilon$ (lies diagonally). Then, $N(\varepsilon)$ is between $l(1/\varepsilon)$and $2l(1/\varepsilon)$. The definition above gives $d = 1$.

Show that the box-counting dimension of a disk is 2.

There are simplifications to the definition (we can have other boxes than squares, other simplifications) that allow easier calculation. It is also sufficient to check $\varepsilon = b_n$ where $\lim_{n\to\infty} b_n = 0$ and $\lim_{n\to\infty} \frac{\ln b_{n+1}}{\ln b_n} = 1$.

Example. We calculate now the box-counting dimension of the middle-third Cantor set. We know that at the step n the set C_n consists of 2^n intervals of length $1/3^n$. It contains the endpoints of all 2^n intervals and they lie 3^{-n} far from each other.

$$\operatorname{boxdim}(C) = \lim_{\varepsilon \to 0} \frac{\ln 2^n}{\ln 3^n} = \frac{\ln 2}{\ln 3}.$$

It should be not surprising that the same result is obtained for the Sierpinski gasket.

There are also other definitions of fractal dimensions (correlation dimension) that allow easier calculation in some cases.

Literatura

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