INVESTICE DO ROZVOJE VZDĚLÁVÁNÍ

Vybrané partie z teorie automatického řízení lineárních systémů

Učební texty k semináři

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TENTO STUDIJNÍ MATERIÁL JE SPOLUFINANCOVÁN EVROPSKÝM SOCIÁLNÍM FONDEM A STÁTNÍM ROZPOČTEM ČESKÉ REPUBLIKY

1. OBSAH

2. Vývoj oboru automatického řízení - Feedback Control: the Origins, the Milestones, and the Trends

Vladimír Kučera

Stav dynamického systému je důsledkem minulosti, a totéž platí o stavu oboru automatického řízení. Jedná se o technický obor, a proto jeho vývoj je silně ovlivňován vnějšími ekonomickými a společenskými faktory. Z těchto předpokladů vychází úvaha o současných trendech rozvoje oboru a jeho budoucnosti.

This plenary reviews the major trends in Feedback Control, identifies emerging challenges for control theory, and forecasts future technological developments in the field.

Realizing that the best way to understand an area is to examine its evolution and the reasons for its existence, a brief history of feedback control is provided first. Ingenious feedback devices can be traced back to the ancient Alexandria. The milestones of this evolution were the flying ball governor of James Watt and its stability analysis by Maxwell, the stability theory of Lyapunov, the conception of three-term or PID controllers, the invention of negative feedback amplifiers, the introduction of Nyquist and Bode charts, and Wiener's cybernetics.

The post war developments included optimal control and filtering, adaptive control, robust control, and hybrid control systems. The computer technology in particular has had a tremendous impact on control theory and its application.

Today, as a result of this evolution, it is possible to implement advanced control methodologies. We have smart sensors and smart actuators. The most dramatic impact of electronic processing occurs in controllers. In times past, computational demands of adaptive, optimal and robust control techniques could not be easily performed. With modern electronics, such operations are possible. Modern electronic implementations are also more immune to aging effects, system noise and disturbances.

The forecast of future technological developments is based on the methods and technologies that emerge in computers, communications, networking, manufacturing, nanoscale science, medicine, and biology. Control theory, on the other hand, is looking for new solutions. There is a strong influence of computer science and engineering. Feedback will be used mostly to stabilize the process and to counteract uncertainties, with other functions achieved by a feedforward.The truly exciting developments in any field will occur where there is a confluence of application drivers and disciplinary development of the subject. Automatic control is no exception. Much attention will have to be paid to education and training. The education must be multidisciplinary, with a focus on teaching general methods rather than vocational skills.

3. METODA POLYNOMIÁLNÍCH ROVNIC V TEORII AUTOMATICKÉHO ŘÍZENÍ - POLYNOMIAL CONTROL: PAST, PRESENT, AND FUTURE

Vladimír Kučera

Metoda polynomiálních rovnic tvoří českou vědeckou školu, která je uznávána po celém světě. Metoda vychází z racionálních přenosů pro lineární systémy, chápe je jako podíl dvou polynomů a návrh regulátoru redukuje na řešení lineárních rovnic pro polynomy. Poskytuje jednoduché výpočetní algoritmy jako alternativu stavových metod návrhu.

3.1. Summary

Polynomial techniques have made important contributions to systems and control theory. Engineers in industry often find polynomial and frequency domain methods easier to use than state equation based techniques. Control theorists show that results obtained in isolation using either approach are in fact closely related.

Polynomial system description provides input-output models for linear systems with rational transfer functions. These models display two important system properties, namely poles and zeros, in a transparent manner. A performance specification in terms of polynomials is natural in many situations; see pole allocation techniques.

A specific control system design technique, called polynomial equation approach, was developed in the 1960s and 1970s. The distinguishing feature of this technique is a reduction of controller synthesis to a solution of linear polynomial equations of specific (Diophantine or Bézout) type.

In most cases, control systems are designed to be stable and to meet additional specifications, such as optimality and robustness. It is therefore natural to design the systems step by step: stabilization first, then the additional specifications each at a time. For this it is obviously necessary to have any and all solutions of the current step available before proceeding any further.

This motivates the need for a parametrization of all controllers that stabilize a given plant. In fact this result has become a key tool for the sequential design paradigm. The additional specifications are met by selecting an appropriate parameter. This is simple, systematic, and transparent. However, the strategy suffers from an excessive grow of the controller order.

This article is a guided tour through the polynomial control system design. The origins of the parametrization of stabilizing controllers, called Youla-Kučera parametrization, are explained. Standard results on reference tracking, disturbance elimination, pole placement, deadbeat control, H_2 control, ℓ_1 control and robust stabilization are summarized. New and exciting applications of the Youla-Kučera parametrization are then discussed: stabilization subject to input constraints, output overshoot reduction, and fixed-order stabilizing controller design.

3.2. Introduction

The majority of control problems can be formulated using the diagram shown in Figure 3.1. Given a plant *S*, determine a controller *R* such that the feedback control system is (asymptotically) stable and satisfies some additional performance specifications such as reference tracking, disturbance attenuation, optimality or robustness.

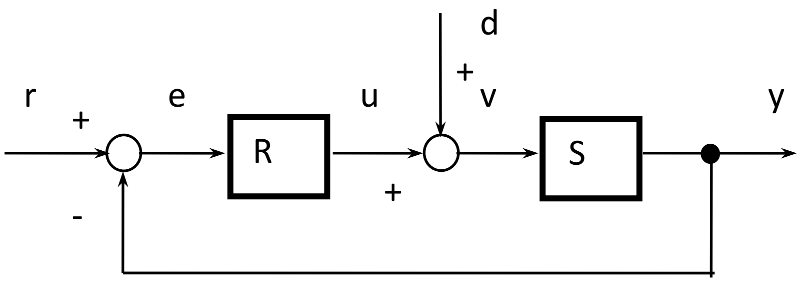

Figure 3.1 Feedback control system

It is natural to separate this task into two steps: (1) stabilization and (2) achievement of additional performance specifications. To do this, all solutions of the first step, i.e. *all controllers that stabilize the given plant*, must be found.

How can one characterize such controllers? Denote H_{sens} the reference-to-error transfer function (sometimes called the sensitivity function) and H_{comp} the disturbance-to-control transfer function (the so called complementary sensitivity function) in the closed-loop control system, namely

$$H_{\mathrm{sens}} = \frac{1}{1+SR}, \quad H_{\mathrm{comp}} = \frac{SR}{1+SR}. \tag{3.1}$$

Now suppose that S can be expressed as the ratio of two coprime polynomials, $S = b/a$, and that the controller has a like form, $R = q/p$. Then the two closed-loop transfer functions can be written as

$$H_{\mathrm{sens}} = a\frac{p}{ap+bq} := aX, \quad H_{\mathrm{comp}} = b\frac{q}{ap+bq} := bY. \tag{3.2}$$

$$H_{\mathrm{sens}} = a\frac{p}{ap+bq} := aX, \quad H_{\mathrm{comp}} = b\frac{q}{ap+bq} := bY.$$

Consequently, if R stabilizes *S* then the rational functions X and Y are bound to be stable. These functions cannot be arbitrary, however, since $H_{\mathrm{sens}} + H_{\mathrm{comp}} = 1$. A stability equation follows [34]

$$aX + bY = 1. \tag{3.3}$$

Any stabilizing controller can be expressed as $R = Y/X$, where X and Y is a stable rational solution pair of the stability equation [21]. This solution can be expressed in parametric form,

$$X = x + bW, \qquad Y = y - aW, \tag{3.4}$$

furnishing in turn an explicit parametrization [38] of all stabilizing controllers

$$R = \frac{y - aW}{x + bW} \tag{3.5}$$

Here *x* and *y* are any polynomials satisfying the equation $ax + by = 1$ while W is a free parameter ranging over the set of stable rational functions.

3.3. Stabilizing Controllers

The intuitive reasoning presented in the introduction will now be made rigorous. Suppose that the plant and the controller are linear time-invariant single-input single-output continuous-time systems with real *rational* transfer functions S and R, respectively. Generalizations will be treated at the end of the article. Further suppose that the state space realizations of S and R are stabilizable and detectable; in case S and R are not proper, the corresponding descriptor realizations are assumed to be also impulse controllable and impulse observable.

3.3.1. *Parametrization of Stabilizing Controllers*

The key result is stated in the form of a theorem; the proof is believed to be the simplest and most comprehensive one available on the subject.

Theorem 3.1

Let $S=\frac{b}{a}$, where a and b are coprime polynomials. Let x and y be two polynomials that satisfy the Bézout equation

$$ax+by=1\,. \tag{3.6}$$

Then the set of all controllers that (asymptotically) stabilize the control system shown in Figure 3.1 is given by

$$R=\frac{y-aW}{x+bW}\,, \tag{3.7}$$

where W is a parameter ranging over the set of stable (i.e., analytic in $\operatorname{Re} s\geq 0$) real rational functions such that $x+bW$ is not identically zero.

Proof consists of three steps.

1) Fist we shall show that if $S=\frac{b}{a}$ and $R=\frac{q}{p}$ are two coprime polynomial fractions, and if c is defined by $c=ap+bq$, then the control system is stable if and only if $1/c$ is a stable rational function.

Indeed, in view of the assumptions on S and R, the control system is stable if and only if the four transfer functions

$$\begin{bmatrix} v \\ y \end{bmatrix} = \frac{1}{1+SR}\begin{bmatrix} 1 & R \\ S & SR \end{bmatrix}\begin{bmatrix} d \\ r \end{bmatrix} = \frac{1}{c}\begin{bmatrix} ap & aq \\ bp & bq \end{bmatrix}\begin{bmatrix} d \\ r \end{bmatrix} \tag{3.8}$$

are all stable. The sufficiency part of the claim is evident: the transfer functions are all seen to be stable. The necessity part is not evident: the denominator c can have zeros in $\operatorname{Re} s \geq 0$ which, conceivably, might cancel in all four numerator polynomials ap, aq, bp and bq. However, this is impossible as the pairs a, b and p, q are both coprime.

2) Further we shall show that a controller R stabilizes the plant $S = b/a$ if and only if it can be expressed in the form $R = Y/X$ for some stable rational solution pair X, Y of the equation $aX + bY = 1$.

Indeed, let X and Y be two stable rational functions that satisfy $aX + bY = 1$. Write X and Y as polynomial fractions, namely $X = p/c$ and $Y = q/c$. Then $ap + bq = c$ and $1/c$ is a stable rational function. Thus $R = Y/X = q/p$ is a stabilizing controller for S. Conversely, suppose that $R = q/p$ stabilizes S and define stable rational functions X and Y by $X = p/c$ and $Y = q/c$, where $c = ap + bq$. Then $aX + bY = 1$.

3) Finally we shall prove that all stable rational solution pairs of the equation $aX + bY = 1$ are given by

$$X = x + bW, \quad Y = y - aW, \tag{3.9}$$

where x, y is a particular polynomial solution pair of this equation and W is a parameter that ranges over the set of stable rational functions.

Indeed, X and Y satisfy the specified equation:

$$a(x + bW) + b(y - aW) = ax + by = 1. \tag{3.10}$$

It remains to show that every stable rational solution pair of the equation has the form shown above for some stable rational function W. We have

$$a(X - x) = b(y - Y). \tag{3.11}$$

Since a and b are coprime, the zeros of a are absorbed in those of $y - Y$ while the zeros of b are absorbed in those of $X - x$. Put $W = (y - Y)/a$, which is a stable rational function. Then $X - x = bW$, and the claim has been proved.

The set of stabilizing controllers for a given plant contains controllers of arbitrarily high order. The set may also contain controllers whose transfer function is not proper (i.e., analytic at $s = \infty$) or is not stable. This is illustrated by the following example.

Example 3.1

Consider an integrator plant $S(s) = 1/s$. The Bézout equation admits a solution $x = 0$, $y = 1$ so that the set of all stabilizing controllers for S is given by

$$R(s) = \frac{1 - sW}{W}$$

for any stable real rational $W \neq 0$.

The parameter

$$W(s) = \frac{1}{s+1}$$

yields $R = 1$, a proportional gain controller. The parameter

$$W(s) = \frac{s}{s^2 + s + 1}$$

results in a proportional-integral controller

$$R(s) = 1 + \frac{1}{s}.$$

Taking $W = 1$ leads to the stabilizing controller $R(s) = 1 - s$. The resulting feedback system is asymptotically stable but it has poles at $s = \infty$. On the other hand, taking

$$W(s) = \frac{s+1}{s^2 + s + 1}$$

yields the stabilizing controller

$$R(s) = \frac{1}{s+1}$$

which itself is stable.

3.3.2. *Discrete-Time Systems*

Theorem 3.1 can be applied to both continuous-time and discrete-time systems. Accordingly, a rational function is defined to be stable if it is analytic either in $\operatorname{Re} s \geq 0$ or in $|z| \geq 1$.

Continuous-time systems can give rise to transfer functions that are not proper. In the case of discrete-time systems, however, additional constraints have to be imposed: the transfer functions S and R are proper (so that the plant and the controller are causal systems) and one of them is strictly proper (so that the closed loop system is causal). The chronology of samples in the control system is usually taken in such a way that S is to be strictly proper.

Proper rational controllers can be obtained by a degree control in the general formula. A better way, however, is to express S as a ratio of two polynomials in z^{-1} and look for polynomials x and y in z^{-1}, solutions of the Bézout equation, such that $x(0) \neq 0$. Then proper rational controllers R correspond to *proper* stable rational parameters W.

Example 3.2

Consider

$$S(z) = \frac{1}{z-1},$$

a sampled version of the integrator plant. Write

$$S(z) = \frac{z^{-1}}{1-z^{-1}}.$$

Then the Bézout equation

$$(1-z^{-1})x + z^{-1}y = 1$$

admits a solution $x=1$, $y=1$. The set of all (proper rational) stabilizing controllers is given by

$$R(z) = \frac{1-(1-z^{-1})W}{1+z^{-1}W}$$

for any proper stable rational W.

3.3.3. *Historical Notes*

The use of polynomials, in one way or another, in feedback control systems design can be traced back to the 1950s [30], [18]. The authors noted that for a closed-loop system to be stable, H_{comp} must absorb the plant unstable zeros. The plant was assumed to be stable; if this assumption were dropped, H_{sens} would have been found to absorb the plant unstable poles. These conditions are equivalent to polynomial divisibility conditions and hence to the stability equation, which appears in [34].

The first attempt to use polynomials in an explicit manner is due to Volgin [37], a student of Tsypkin. He obtained a solution of the pole placement problem through the solution of a polynomial equation, known as the pole placement equation. In the early 1970s, Åström [3] published a polynomial equation solution to the minimum variance control problem. The solution was limited to minimum phase plants; a general solution was subsequently obtained by Peterka [31]. The ultimate book that presents the polynomial equation approach to multi-input multi-output control system design is [23].

The underlying problem in any control system design is that of stability. It is logical to design the control system step by step: stabilization first, then the additional performance specifications. To do this, we need to know any and all stabilizing controllers for the given plant.

This problem was first addressed and solved by Kučera [21] in single-input single-output discrete-time systems. A generalization of this result to multi-input multi-output systems was published in [22], [23]. At the same time, and entirely independently, an explicit parametrization of all stabilizing controllers for continuous-time plants was obtained by Youla *et al.* [38], [39] in the process of ensuring stability for linear-quadratic control systems.

It took decades to appreciate the importance of the result and come up with applications. The milestones were the observations by Desoer *et al*. [6] and Vidyasagar [36] that the polynomial fraction approach can be extended to linear systems with non-rational transfer functions, as well as the result by Hammer [10] showing that the approach is applicable to a broad class of non-linear plants. The parametrization was labeled in [1] as the *Youla-Kučera* *parametrization*. This result launched an entirely new area of research and has ultimately become a new paradigm for control system design.

3.4. Additional Performance Specifications

Theorem 3.1 shows that there is a simple formula that generates all the stabilizing controllers for a given plant. Using this formula, we can obtain a parametrization of all stable closed-loop transfer functions that can be obtained by stabilizing a given plant. The bonus is that the parametrization is *affine* in the free parameter W. In contrast, the controller R appears in a nonlinear fashion:

$$\begin{bmatrix} v \\ y \end{bmatrix} = \frac{1}{1+SR}\begin{bmatrix} 1 & R \\ S & SR \end{bmatrix}\begin{bmatrix} d \\ r \end{bmatrix} = \begin{bmatrix} a(x+bW) & a(y-aW) \\ b(x+bW) & b(y-aW) \end{bmatrix}\begin{bmatrix} d \\ r \end{bmatrix}. \tag{3.12}$$

As R and W are in a one-to-one correspondence, it is convenient to use W in lieu of R in the design process and calculate R subsequently. Thus the parametrization of all stabilizing controllers makes it possible to separate the design process into two steps: the determination of all stabilizing controllers and the selection of the parameter that achieves the remaining design specifications. The extra benefit is that both tasks are linear.

3.4.1. *Asymptotic Properties*

Asymptotic properties of control systems can easily be accommodated in the sequential design procedure. These include the elimination of an offset due to step references, the ability of system output to follow a class of reference signals, or the asymptotic elimination of specific disturbances [7].

In Figure 3.1, asymptotic *reference tracking* means that the output y follows the reference r as time approaches infinity, which is to say that the error e approaches zero for large times. On the other hand, we speak of asymptotic *disturbance elimination* if the effect of the disturbance d decreases at the output y for increasing time. In terms of Laplace transforms, $e = H_{\text{sens}} r$ and $y = SH_{\text{sens}} d$ are to be *stable* rational functions.

Example 3.3

Consider the plant $S(s) = 1/(s+1)$. The Bézout equation admits a solution $x = 0$, $y = 1$. The set of all stabilizing controllers for S is

$$R(s) = \frac{1-(s+1)W}{W}$$

for any stable real rational $W \neq 0$. The achievable sensitivity transfer functions are $H_{\text{sens}} = (s+1)W$.

To track a step reference, $r = 1/s$, we must take $W = sW_1$ for any stable rational $W_1 \neq 0$. To eliminate a sinusoidal disturbance, $d = s/(s^2+\omega^2)$, we constrain the parameter as $W = (s^2+\omega^2)W_2$ for any stable rational $W_2 \neq 0$. To meet both requirements, we simply take $W = (s^2+\omega^2)W_3$ for any stable rational $W_3 \neq 0$, say $W = s(s^2+\omega^2)/(s+1)^4$.

The resulting controller is

$$R(s) = \frac{3s^3+(6-\omega^2)s^2+(4-\omega^2)s+1}{s(s^2+\omega^2)}.$$

The controller obtained in Example 3.3 demonstrates the internal model principle: the unstable modes to be followed or eliminated must be generated by the controller unless they are present in the plant.

3.4.2. *Pole Placement*

The requirement of stability places all closed-loop system poles within the left half-plane $\operatorname{Re} s < 0$. Very often, however, we wish to allocate the poles to a specific region of the half-plane or to achieve specific pole positions. Given a plant $S = b/a$, the set of all the stabilizing controllers for S is $R = \frac{y-aW}{x+bW}$ where x, y are polynomials such that $ax+by=1$and W is a free stable rational parameter. Let $W = w/d$ for a stable polynomial d. Then $R = \frac{dy-aw}{dx+bw} := \frac{q}{p}$ and the closed-loop system poles (assuming that S and R are both controllable and observable) are given by $ap+bq = d(ax+by) = d$. Thus W parametrizes all stabilizing controllers for S, the denominator polynomial d of W specifies the positions of the control system poles, and the numerator polynomial w od W

represents the remaining degrees of freedom, i.e., parametrizes all stabilizing controllers that assign the specified poles.

Example 3.4

Consider the plant $S(s)=1/(s-1)$ and the set of stabilizing controllers for S:

$$R(s)=\frac{1-(s-1)W}{W}.$$

Let the desired pole locations be given by the polynomial $d=s^2+\delta_1 s+\delta_0$. This is achieved by putting $W=w/d$ for an arbitrary numerator polynomial $w\neq 0$.

It is to be noted that *d* specifies the poles at *finite* positions only. Poles at $s=\infty$ will occur whenever the control system has order higher than 2. The order of the plant is one, so only the controllers of order one will not generate infinite poles. These controllers correspond to the choice $w=s+\omega_0$ for any real ω_0.

3.4.3. *Deadbeat Control*

Deadbeat control is a typical discrete-time control strategy. Given a plant with transfer function S, written in the form of a coprime fraction of two polynomials in z^{-1}, $S=b/a$. The task is to determine a controller R that stabilizes the control system of Figure 3.1 and endows its four transfer functions H_{sens}, SH_{sens} and H_{comp}, $S^{-1}H_{\text{comp}}$ with the *finite impulse response* property, that is to say, the corresponding impulse responses vanish in a finite time [27].

In a stabilized control system, the achievable sensitivity and complementary sensitivity functions can be parametrized as follows:

$$H_{\text{sens}}=a(x+bW),\quad H_{\text{comp}}=b(y-aW). \tag{3.13}$$

Similarly,

$$SH_{\text{sens}}=b(x+bW),\quad S^{-1}H_{\text{comp}}=a(y-aW). \tag{3.14}$$

To have the finite impulse response property, the four transfer functions must be *polynomials* in z^{-1}. Since a and b are coprime, this is the case if and only if *W* is a polynomial in z^{-1}.

Consequently, deadbeat controller assigns the pole polynomial $d=1$ in the indeterminate z^{-1}, i.e., all closed-loop system poles are located at the point $z=0$.

For the impulse responses to be finite and *as short as possible*, we simply select W so as to minimize the degrees of H_{sens} and H_{comp}. This corresponds to taking the *least degree solution* pair x, y of the Bézout equation $ax+by=1$ and setting $W=0$.

Example 3.5

Consider $S(z')=z^{-1}/(1-z^{-1})$, a sampled version of the integrator plant. Then the least degree solution of the Bézout equation

$$(1-z^{-1})x+z^{-1}y=1$$

is $x=1$, $y=1$ and the set of all stabilizing controllers was found in Example 3.2 to be

$$R(z)=\frac{1-(1-z^{-1})W}{1+z^{-1}W}$$

for any proper stable rational W.

The resulting sensitivity and complementary sensitivity functions are parametrized as

$$H_{\text{sens}}(z)=1-z^{-1}+z^{-1}(1-z^{-1})W$$
$$H_{\text{comp}}(z)=z^{-1}-z^{-1}(1-z^{-1})W$$

and similarly

$$SH_{\text{sens}}(z)=z^{-1}+z^{-2}W$$
$$S^{-1}H_{\text{comp}}(z)=1-z^{-1}-(1-z^{-1})^2W.$$

These functions are seen to be polynomials in z^{-1} if and only if W is so. The shortest impulse responses are achieved for $W=0$ $H_{\text{sens}}=1-z^{-1}$, $SH_{\text{sens}}=z^{-1}$ and $H_{\text{comp}}=z^{-1}$, $S^{-1}H_{\text{comp}}=1-z^{-1}$. The resulting deadbeat controller is $R=1$.

3.4.4. H_2 Optimal Control

The sequential design procedure will be further illustrated on the design of *linear-quadratic optimal* controllers. Given a plant with transfer function $S = b/a$, the task is to find a controller that stabilizes the control system of Figure 3.1 while minimizing the H_2 norm of some closed-loop transfer function, say of the complementary sensitivity function H_{comp}, which is defined [7], [40] by

$$\left\| H_{\text{comp}} \right\|_2 = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \left| H_{\text{comp}}(j\omega) \right|^2 d\omega \right)^{\frac{1}{2}}. \tag{3.15}$$

The set of sensitivity functions that can be achieved in the stabilized control system is

$$H_{\text{comp}} = b(y - aW), \tag{3.16}$$

where W is a free stable rational parameter. The parameter will be selected so as to minimize the norm of H_{comp}.

Let $\alpha\beta$ be a polynomial defined by keeping the stable (in $\operatorname{Re} s < 0$) zeros of ab while replacing the unstable (in $\operatorname{Re} s \geq 0$) ones with their negative values. Then $ab/\alpha\beta$ is inner (or all-pass) and

$$\left\| H_{\text{comp}} \right\|_2 = \left\| \frac{\alpha\beta}{ab} H_{\text{comp}} \right\|_2 = \left\| \frac{\alpha y \beta}{a} - \alpha W \beta \right\|_2. \tag{3.17}$$

Consider the decomposition

$$\frac{\alpha y \beta}{a} = p + \frac{q}{a} \tag{3.18}$$

with p polynomial and q/a strictly proper. With this decomposition,

$$\left\| H_{\text{comp}} \right\|_2^2 = \left\| \frac{q}{a} \right\|_2^2 + \left\| p - \alpha W \beta \right\|_2^2 \tag{3.19}$$

because q/a and $p - \alpha W \beta$ are orthogonal and thus the cross-terms contribute nothing to the norm. The last expression is a complete square whose first part is independent of W. Hence the minimizing parameter is $W = p/\alpha\beta$ and if it is indeed stable and admissible, it defines the unique optimal controller. Otherwise, no optimal controller exists.

The consequent minimum norm equals

$$\min_W \left\| H_{\text{comp}} \right\|_2 = \left\| \frac{q}{a} \right\|_2 . \tag{3.20}$$

It is easy to see that the (finite) pole positions of the H_2 optimal control system are given by the pole polynomial $d = \alpha\beta$.

Example 3.6

To illustrate, consider the plant $S(s) = 1/(s-1)$. The class of all stabilizing controllers for S was found in Example 3.4, namely

$$R(s) = \frac{1-(s-1)W}{W}$$

for a free stable rational parameter $W \neq 0$. The complementary sensitivity transfer function is

$$H_{\text{comp}}(s) = 1-(s-1)W .$$

Now $\alpha = s+1$, $\beta = 1$ and the polynomial part of

$$\frac{\alpha y \beta}{a} = \frac{s+1}{s-1} = 1 + \frac{2}{s-1}$$

is $p = 1$. Thus H_{comp} attains minimum H_2 norm for

$$W = \frac{1}{s+1}$$

and the corresponding optimal controller is $R(s) = 2$.

The optimal complementary sensitivity function is

$$H_{\text{comp}} = \frac{2}{s+1}$$

and $\left\| H_{\text{sens}} \right\|_2 = \sqrt{2}$.

3.4.5. ℓ_1 Optimal Control

The H_2 norm minimization is appropriate for systems excited by finite energy signals. When the exogenous signals persist, a more relevant norm to measure system performance is the L_1 norm (for continuous-time systems) or the ℓ_1 norm (for discrete-time systems). The discrete-time case is much easier.

The problem is posed as follows. Given a *discrete-time* plant $S = b/a$, find a controller R that stabilizes the control system shown in Figure 3.1 while giving rise to some closed-loop transfer function, say H_{sens}, whose impulse response h_{sens} is of minimal ℓ_1 norm.

The ℓ_1 norm of the sequence $h_{\text{sens}} = (h_0, h_1, h_2, \cdots)$ is defined [7] as

$$\| h_{\text{sens}} \|_{\ell_1} = \sum_{i=0}^{\infty} h_i \,. \tag{3.21}$$

Since $H_{\text{sens}}(z) = h_0 + h_1 z^{-1} + h_2 z^{-2} + \cdots$, the ℓ_1 norm of sequences implies a 1-norm of the corresponding z-transforms, namely

$$\| H_{\text{sens}} \|_1 = \| h_{\text{sens}} \|_{\ell_1} . \tag{3.22}$$

The set of sensitivity functions that can be achieved in the stabilized control system of Figure 3.1 is

$$H_{\text{sens}} = ax + abW, \tag{3.23}$$

where W is a free stable rational parameter. The task is to select W so as to minimize the 1-norm of H_{sens}. This minimization problem is solvable if and only if a and b have no zeros on the unit circle [5]. The optimal sensitivity function H_{sens} is not unique but has a *finite impulse response* property [17].

In view of this property, we express all transfer functions as ratios of polynomials in z^{-1}. Perform the stable-unstable factorizations $a = a^+ a^-$ and $b = b^+ b^-$, where a^- and b^- absorb all the zeros of a and b, respectively, in the open unit disc $|z^{-1}| < 1$. Then H_{sens} is a polynomial in z^{-1} if and only if W has the form

$$W = \frac{w}{a^+b^+},\qquad (3.24)$$

where *w* is a free polynomial in z^{-1}. Indeed,

$$H_{\text{sens}} = ax + a^-b^-w,\qquad (3.25)$$

and the 1-norm minimization of H_{sens} is equivalent to a finite linear program for the coefficients of w.

The pole positions of the ℓ_1 optimal control system are given by the pole polynomial $d = a^+b^+$ in the indeterminate z^{-1}, duly completed by poles at $z = 0$.

Example 3.7

Consider a plant with the transfer function

$$S = \frac{1-1.5z}{(z-2)^2} = z^{-1}\frac{z^{-1}-1.5}{(1-2z^{-1})^2}.$$

The Bézout equation $ax + by = 1$ admits a solution $x = 1-0.5z^{-1}$, $y = -3+2z^{-1}$ and the set of stabilizing controllers is given by the formula

$$R = \frac{-3+2z^{-1}-(1-2z^{-1})^2W}{1-0.5z^{-1}+z^{-1}(z^{-1}-1.5)W}$$

for any stable rational W.

The set of achievable sensitivity functions is

$$H_{\text{sens}} = (1-2z^{-1})^2(1-0.5z^{-1}) + z^{-1}(1-2z^{-1})^2(z^{-1}-1.5)W$$

and those which are *polynomials* in z^{-1} are

$$H_{\text{sens}} = (1-2z^{-1})^2(1-0.5z^{-1}) + z^{-1}(1-2z^{-1})^2 w,$$

where *w* is the numerator polynomial in z^{-1} of

$$W = \frac{w}{z^{-1}-1.5}.$$

An upper bound for the degree of w, as follows from a result obtained in [5], is 2. The linear program:

minimize $t = r_1 + r_2 + r_3 + r_4 + r_5$

subject to $-r_i \le h_i \le r_i$ and $r_i \ge 0$, $i = 1,2,\cdots 5$

where

$$\begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \end{bmatrix} = \begin{bmatrix} -4.5 \\ 6 \\ -2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 4 & -4 & 1 \\ 0 & 4 & -4 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \\ w_2 \end{bmatrix}$$

then returns

$$w_0 = 1.5, \quad w_1 = 0, \quad w_2 = 0$$

so that

$$W = \frac{1.5}{z^{-1} - 1.5}.$$

The optimal controller is

$$R = \frac{3 - 4z^{-1}}{(1 + z^{-1})(z^{-1} - 1.5)},$$

the corresponding optimal sensitivity function is

$$H_{\text{sens}} = 1 - 3z^{-1} + 4z^{-3},$$

and $\| h_{\text{sens}} \|_{\ell_1} = 8$.

It is to be noted that R is *not* a deadbeat controller because SH_{sens} is not a polynomial. Indeed, only polynomial parameters W result in deadbeat controllers

3.4.6. *Robust Stabilization*

The notion of robust stability addresses stabilization of plants subject to modeling errors, when the actual plant may differ from the nominal model, using a fixed controller. The ultimate goal is to stabilize the actual plant. The actual plant is unknown, however, so the best one can do is to stabilize a large enough set of plants.

Thus the basis technique to model plant uncertainty is to model the plant as belonging to a set. Such a set can be either structured – for example, there is a finite number of uncertain parameters – or unstructured – the frequency response lies in a set in the complex plane for every frequency. The unstructured uncertainty model is more important for several reasons. On the one hand, it is well suited to represent high-frequency modeling errors, which are generically present and caused by such effects as infinite-dimensional electromechanical resonance, transport delays, and diffusion processes. On the other hand, the unstructured model of uncertainty leads to a simple and useful design theory.

The unstructured set of plants is usually constructed as a neighborhood of the nominal plant, with the uncertainty represented by additive or multiplicative perturbations [5], [40]. The size of the neighborhood is measured by a suitable norm, most common being the H_∞ norm that is defined for any rational function G analytic on the imaginary axis as

$$\|G\|_\infty = \sup_\omega |G(j\omega)| . \tag{3.26}$$

This section will illustrate the design for *robust stability under unstructured norm-bounded multiplicative perturbations*. Consider a nominal plant with transfer function S and its neighborhood S_Δ defined by

$$S_\Delta := (1 + F\Delta) S , \tag{3.27}$$

where F is a fixed stable rational function and Δ is a variable stable rational function such that $\|\Delta\|_\infty \le 1$. The idea behind this uncertainty model is that $F\Delta$ is the normalized plant perturbation away from 1:

$$\frac{S_\Delta}{S} - 1 = V\Delta . \tag{3.28}$$

Hence if $\|\Delta\|_\infty \le 1$, then for all frequencies ω

$$\left| \frac{S_\Delta(j\omega)}{S(j\omega)} - 1 \right| \le |F(j\omega)| \tag{3.29}$$

so that $|F(j\omega)|$ provides the uncertainty profile while Δ accounts for phase uncertainty.

Now suppose that R is a controller that stabilizes the nominal plant S. Consequently, R will stabilize the entire family of plants S_Δ if and only if [5], [40]

$$\|H_{\text{comp}} F\|_\infty < 1 \, . \tag{3.30}$$

This is a necessary and sufficient condition for robust stabilization of the nominal plant S.

The set of all stabilizing controllers for $S = b/a$ is described by the formula

$$R = \frac{y - aW}{x + bW} \, , \tag{3.31}$$

where $ax + by = 1$ and W is a free stable rational parameter. The robust stability condition then reads

$$\|b(y - aW)F\|_\infty < 1 \, . \tag{3.32}$$

Any stable rational W that satisfies this inequality then defines a robustly stabilizing controller R for S. In case W actually minimizes the norm one obtains the best robustly stabilizing controller.

Example 3.8

Consider a plant with the transfer function [26]

$$S_\tau(s) = \frac{s+1}{s-1} \mathrm{e}^{-\tau s}$$

where the time delay τ is known only to the extent that it lies in the interval $0 \le \tau \le 0.2$. The task is to find a controller that stabilizes the uncertain plant S_τ. The time-delay factor $\mathrm{e}^{-\tau s}$ can be treated as a multiplicative perturbation of the nominal plant

$$S(s) = \frac{s+1}{s-1}$$

by embedding S_τ in the family

$$S_\Delta = (1 + F\Delta)S \text{ ,}$$

where Δ ranges over the set of stable rational functions such that $\|\Delta\|_\infty \le 1$. To do this, F should be chosen so that the normalized perturbation satisfies

$$\left| \frac{S_\Delta(j\omega)}{S(j\omega)} - 1 \right| = \left| e^{-j\omega\tau} - 1 \right| \le |F(j\omega)|$$

for all ω and τ. A little time with the Bode magnitude plot [5] shows that a suitable uncertainty profile is

$$F(s) = \frac{3s+1}{s+9} .$$

Figure 3.2 is the Bode magnitude plot of this F and $e^{-\tau s} - 1$ for $\tau = 0.2$, the worst value.

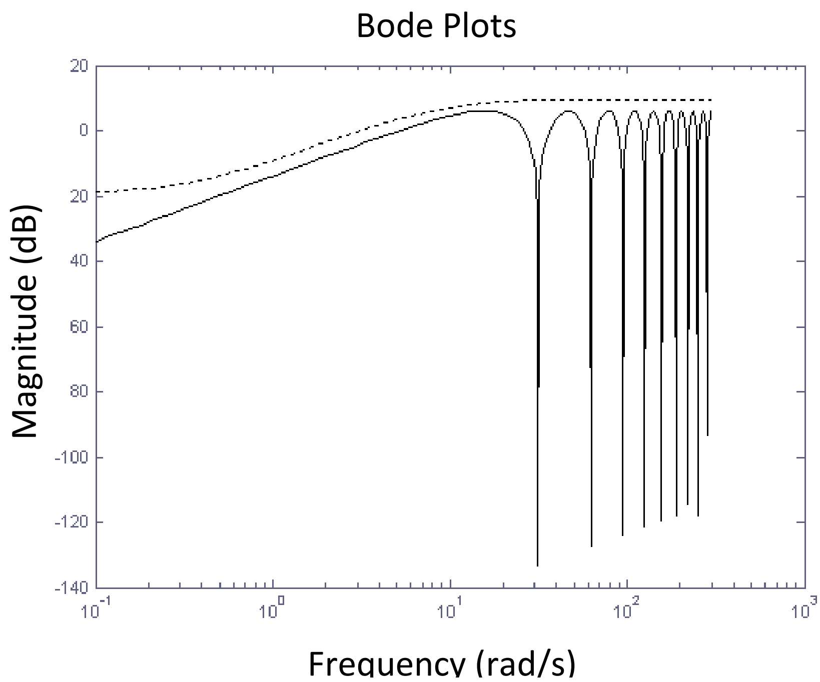

Figure 3.2 Bode plots of F (dotted) and $e^{-0.2s} - 1$ (solid)

The task of stabilizing the uncertain plant S_τ is thus replaced by that of stabilizing every element in the set S_Δ, that is to say, by robustly stabilizing the nominal plant S with respect to the multiplicative perturbations defined by F.

The set of all stabilizing controllers for S is found to be

$$R(s) = \frac{\frac{1}{2} - (s-1)W}{-\frac{1}{2} + (s+1)W}$$

where $W \neq 2(s+1)$ is any stable rational parameter. The robust stability condition reads

$$\| P - QW \|_\infty < 1$$

where

$$P(s) = \frac{1}{2}(s+1)\frac{3s+1}{s+9}, \quad Q(s) = (s-1)(s+1)\frac{3s+1}{s+9}.$$

Since Q has one unstable zero at $s=1$, it follows from the maximum modulus theorem [5] that the minimum of the H_∞ norm taken over all stable rational functions W is $P(1) = 2/5$ and this minimum is achieved for

$$W(s) = \frac{P(s) - P(1)}{Q(s)} = \frac{1}{10}\frac{15s+31}{(s+1)(3s+1)}.$$

Thus the robust stability condition is satisfied and the corresponding best robustly stabilizing controller is

$$R(s) = \frac{2}{13}\frac{s+9}{s+1}.$$

3.5. Advanced Applications

The step-by-step design paradigm has found numerous applications in the literature. Although the idea is 30 years old, it is still a subject of current interest. This will be demonstrated by presenting several advanced applications that address control problems difficult to solve otherwise, or provide alternative solutions with attractive features.

3.5.1. *Stabilization Subject to Input Constraints*

Most plants have inputs that are subject to hard limits on the range of variations that can be achieved. The effects of actuator saturation on a control system are poor performance and/or instability. Stabilization subject to input constraints can be formulated either as a local stabilization, when saturation is avoided for a set of initial states and the control system behaves as a linear one, or as a global stabilization, when saturation is allowed to occur and the control system is nonlinear.

Consider the saturation avoidance approach [15]. Given a discrete-time plant

$$y = Su + Tx_0 \tag{3.33}$$

with x0 the initial state and with the input $u = u_0 + u_1 z^{-1} + u_2 z^{-2} + \ldots$ subject to the constraints $-u^- \le u_k \le u^+, \quad k = 0,1,2,\ldots$

where u^+ and u^- are positive constants. The task is to find a controller of the form (zero initial state w_0 assumed)

$$u = -Ry + Qw_0 \tag{3.34}$$

such that the control system shown in Figure 3.3 is locally asymptotically stable for any initial state x_0 of the plant within a given polyhedron $P_F = \{x : Fx \le f\}$, where F is a matrix and f is a vector.

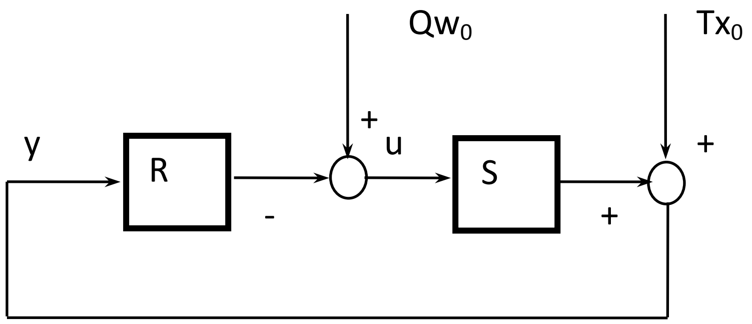

Figure 3.3 Control system with initial states

Denote $S = b/a$ and $T = c/a$ the polynomial fraction representation of the plant. The control sequences in a stabilized closed-loop system are parametrized as

$u = -c(y - aW)x_0$.

Taking W in the form of a power series around the point $z = \infty$

$$W = p_0 + p_1 z^{-1} + p_2 z^{-2} + \dots \tag{3.35}$$

shows that the control sequence is an affine function of the parameters p_0, p_1, p_2, ... of the form

$$u_k = G_k(p_0, p_1, \dots)x_0, \quad k = 0, 1, 2, \dots \tag{3.36}$$

and satisfies the given constraint if x_0 belongs to the polyhedron $P_G = \{x : G(p_1, p_2, \dots)x \le g\}$ generated by

$$G(p_0, p_1, \dots) = \begin{bmatrix} G_0(p_0, p_1, \dots) \\ -G_0(p_0, p_1, \dots) \\ G_1(p_0, p_1, \dots) \\ -G_1(p_0, p_1, \dots) \\ \vdots \end{bmatrix}, \quad g = \begin{bmatrix} u^+ \\ u^- \\ u^+ \\ u^- \\ \vdots \end{bmatrix}. \tag{3.37}$$

Now x_0 is in P_F , so that P_F must be contained in P_G. Applying the Farkas lemma [15], one concludes that the stabilization problem has a solution if and only if there exists a matrix P with non-negative entries and real numbers p_0, p_1, p_2, ... such that

$$PF = G(p_0, p_1, \dots), \quad Pf \le g. \tag{3.38}$$

This is a linear program for P and p_0, p_1, p_2, ... The stabilizing controller is then obtained by putting

$$W = p_0 + p_1 z^{-1} + p_2 z^{-2} + \dots. \tag{3.39}$$

If the power series W is approximated by a *polynomial* in z^{-1}, then the program has a finite dimension.

Example 3.9

Consider the plant described by the input-output and state-output transfer functions

$$S(z)=\frac{z^{-1}}{1-2z^{-1}}, \quad T(z)=\frac{2}{1-2z^{-1}}$$

The plant input is constrained as

$$-1 \le u_k \le 1, \quad k=0,1,2,\ldots$$

and the initial state x_0 belongs to the polyhedron

$$P_F : \begin{bmatrix} 1 \\ -1 \end{bmatrix} x_0 \le \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} \qquad \text{(or } |x_0| \le 1/3\text{).}$$

The set of stabilizing controllers is found to be

$$R(z)=\frac{2-(1-2z^{-1})W}{1+z^{-1}W}$$

for a free, proper stable rational parameter W. The corresponding control sequence is

$$u(z)=[-4+2(1-2z^{-1})W]x_0 .$$

Now start with $W=0$ and check whether the resulting linear program for P is feasible:

$$P\begin{bmatrix} 1 \\ -1 \end{bmatrix}=\begin{bmatrix} -4 \\ 4 \end{bmatrix}, \qquad P\begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} \le \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

It is not, hence no controller of order 0 stabilizes the plant.

Proceed by choosing $W=p_0$ and check whether the resulting linear program for p_0 and P is feasible:

$$P\begin{bmatrix} 1 \\ -1 \end{bmatrix}=\begin{bmatrix} -4+2p_0 \\ 4-2p_0 \\ -4p_0 \\ 4p_0 \end{bmatrix}, \qquad P\begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} \le \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

It is, and the solution

$$p_0 = \frac{2}{3}, \quad P = \frac{1}{3}\begin{bmatrix} 0 & 8 \\ 8 & 0 \\ 0 & 8 \\ 8 & 0 \end{bmatrix}$$

furnishes the stabilizing controller

$$R(z) = \frac{4+4z^{-1}}{3+2z^{-1}}.$$

The actual polyhedron of stabilizable initial states is

$$P_G : \frac{1}{3}\begin{bmatrix} -8 \\ 8 \\ -8 \\ 8 \end{bmatrix} x_0 \le \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad \text{(or } |x_0| \le 3/8\text{)}$$

and it includes P_F as a proper subset.

Note that the closed-loop control system features the finite impulse response property. Selecting a polynomial parameter W implies that the closed-loop poles are all at the origin.

3.5.2. *Input and Output Shaping*

In addition to actuator amplitude or rate limits, control system design often has to take into account output signal overshoot or undershoot, trajectory planning constraints and other time-domain specifications.

As seen in the preceding section, such constraints are easy to handle in discrete-time systems. The *z*-transform provides a simple direct relationship between the signals and their transforms:

$$(u_0,\ u_1,\ u_2, \ldots) \quad \Leftrightarrow \quad u_0 + u_1 z^{-1} + u_2 z^{-2} + \ldots \tag{3.40}$$

However, this is not true for the Laplace transform applied in continuous-time systems. The best parallel we can make [16] is to assign distinct negative real poles (rather than placing them all in the origin) and express signals as polynomials in the corresponding exponential modes.

Given a plant $S = b/a$, we are seeking in Figure 3.1 a stabilizing controller $R = q/p$ such that the output y asymptotically follows a reference r while the time-domain constraints

$$u_{\min} \le u(t) \le u_{\max}\ , \quad y_{\min} \le y(t) \le y_{\max} \tag{3.41}$$

are satisfied for all $t \ge 0$, where $u_{\min}$, $u_{\max}$, $y_{\min}$ and $y_{\max}$ are given real numbers. We assume that S is strictly proper and that R is proper so as to avoid impulsive modes.

Assign distinct negative *integer* poles s_i

$$ap + bq = d\ := \Pi_i (s - s_i)\,. \tag{3.42}$$

Then signals in the closed-loop system are sums of the corresponding decaying exponential modes,

$$u(t) = \Sigma_i u_i \mathrm{e}^{-s_i t}, \quad y(t) = \Sigma_i y_i \mathrm{e}^{-s_i t}\,. \tag{3.43}$$

Let g be the greatest common divisor of the poles s_i, so that $s_i = k_i g$ for some integers k_i. The signals can now be expressed as polynomials in the indeterminate $\lambda = \mathrm{e}^{-gt}$, namely

$$u(t) = \Sigma_i u_i \lambda^{k_i}, \quad y(t) = \Sigma_i y_i \lambda^{k_i}\,. \tag{3.44}$$

When time t increases from 0 to ∞, indeterminate λ decreases from 1 to 0 and the time constraints become the polynomial bound constraints

$$u_{\min} \le u(\lambda) \le u_{\max},\ y_{\min} \le y(\lambda) \le y_{\max} \tag{3.45}$$

or, equivalently, the polynomial non-negativity constraints

$$\begin{aligned} &u(\lambda) - u_{\min} \ge 0, \quad -u(\lambda) + u_{\max} \ge 0, \\ &y(\lambda) - y_{\min} \ge 0, \quad -y(\lambda) + y_{\max} \ge 0, \end{aligned} \tag{3.46}$$

along the interval $\lambda \in [0,1]$.

A polynomial non-negativity constraint

$$p(\lambda) = \Sigma_{i=0}^{n} p_i \lambda^i \ge 0, \quad \forall \lambda \in [\lambda_{\min}, \lambda_{\max}] \tag{3.47}$$

is equivalent [13] to the existence of real symmetric matrices $P_{\min}$, $P_{\max}$ of size n + 1 satisfying the linear matrix inequality constraints

$$\begin{gathered} p_i = \mathrm{trace}[P_{\min}(H_{i-1} - \lambda_{\min} H_i)] + \mathrm{trace}[P_{\max}(\lambda_{\max} H_i - H_{i-1})], \quad i = 0, 1, \ldots, n \\ P_{\min} \geq 0, \; P_{\max} \geq 0 \end{gathered}, \tag{3.48}$$

where H_i is the basis Hankel matrix with ones along the $(i+1)$th anti-diagonal and zeros elsewhere.

Now all proper rational controllers R that assign the pole polynomial $d = \prod_i (s - s_i)$ are parametrized by a polynomial w of appropriate degree, see Section 3.2. The coefficients of w are our design parameters and they appear in the coefficients u_i, y_i of the closed-loop signals in an affine manner. Therefore the linear matrix inequalities are convex in the design parameters.

Example 3.10

Given the plant [16]

$$S = \frac{s + 0.5}{s(s-2)},$$

the stabilizing controller

$$R = \frac{384s + 240}{s^3 + 17s^2 + 119s + 79}$$

assigns the closed-loop system poles at − 1, − 2, − 3, − 4, − 5 while ensuring asymptotic step reference tracking. Despite the poles being negative real, the step response features an unacceptable overshoot of 140 % due to system zeros, see Figure 3.4.

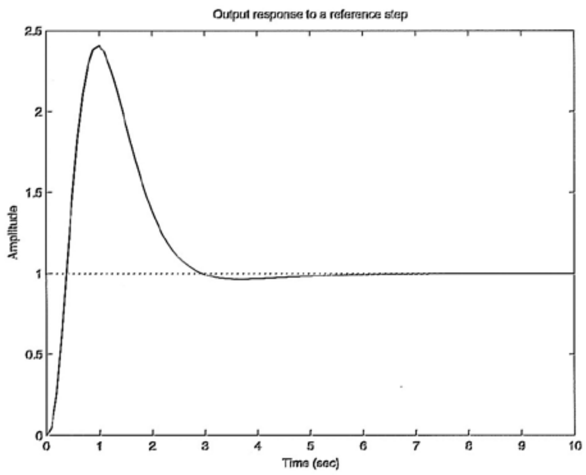

Figure 3.4 Step response with unacceptable overshoot

The set of all proper rational controllers that assign the above poles is given by

$$R(s) = \frac{384s + 240 - s(s-2)w}{s^3 + 17s^2 + 119s + 79 + (s+0.5)w}$$

where $w = w_0 + w_1 s$ is a free polynomial of degree at most 1. The closed-loop responses to a step input are affine in w,

$$y(s) = \frac{384s^2 + 432s + 120 - s(s^2 - 1.5s - 1)w}{s(s+1)(s+2)(s+3)(s+4)(s+5)},$$

and correspond to a sum of decaying exponential modes in the time domain,

$$y(t) = \Sigma_{i=0}^{5} y_i \mathrm{e}^{-it}$$

or to a polynomial

$$y(\lambda) = \Sigma_{i=0}^{5} y_i \lambda^i$$

in the indeterminate $\lambda = \mathrm{e}^{-t}$. The coefficients y_i are affine functions of w_0 and w_1.

Suppose the desired maximum overshoot is 20 %. This specification translates as $y(t) \le 1.2 y_0$ and is equivalent to the polynomial non-negativity constraint

$$p(\lambda) = 0.2y_0 - y_1\lambda - y_2\lambda^2 - y_3\lambda^3 - y_4\lambda^4 - y_5\lambda^5 \ge 0$$

along the interval $\lambda \in [0,1]$. This is in turn equivalent to the linear matrix inequality problem for w_0 and w_1. A standard solver returns

$$w(s) = -100.36 - 12.27s$$

and the closed-loop response shown in Figure 3.5.

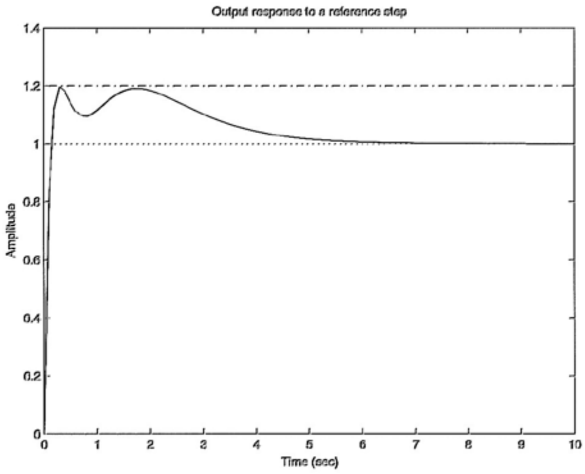

Figure 3.5 Step response with reduced overshoot

Our design conditions are necessary and sufficient as soon as we fix the poles to be assigned. So it may happen that the constraints are not satisfied with a given choice of poles, whereas they could be satisfied with another choice.

3.5.3. *Fixed-Order Stabilizing Controllers*

A weakness of the sequential design based on the Youla-Kučera parametrization is that each performance specification beyond stability may *increase the order* of the controller.

The degree control in the stable rational parameter $W = w/d$ is difficult. If d is fixed, all closed-loop transfer functions are affine in w but the order of w increases with each additional performance specification. If d is not fixed, we have a greater flexibility but we run into difficulties as the set of stable polynomials is not convex in the space of coefficients.

The difficulty was resolved in [12] by providing a *convex inner approximation* of the non-convex stability domain [14] in the space of polynomial coefficients.

This approximation is parametrized by a given stable polynomial, referred to as the *central* polynomial, as explained below.

Given a fixed stable "central" polynomial $c(s)$ of degree n, polynomial $d(s)$ of degree n is stable if there exists a real symmetric matrix Q of size n solving the linear matrix inequality

$$H_c(d,Q) = c^{\mathrm{T}}d + d^{\mathrm{T}}c - \varepsilon c^{\mathrm{T}}c + \Pi_1^{\mathrm{T}}Q\Pi_2 + \Pi_2^{\mathrm{T}}Q\Pi_1 \geq 0, \tag{3.49}$$

where

$$\Pi_1 = \begin{bmatrix} 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \end{bmatrix}, \quad \Pi_2 = \begin{bmatrix} 0 & 1 & & \\ \vdots & & \ddots & \vdots \\ 0 & & & 1 \end{bmatrix}$$

are projection matrices, c and d are the coefficient vectors of the polynomials $c(s)$ and $d(s)$, and ε is an arbitrarily small positive scalar.

The interpretation of this result is as follows: as soon as polynomial c is fixed, we obtain a sufficient linear matrix inequality condition for stability of polynomial d. Therefore,

$$H_c = \{d : \exists Q : H_c(d,Q) \geq 0\} \tag{3.50}$$

is a convex inner approximation of the (generally non-convex) stability domain in the space of polynomial coefficients.

Let us now show how to design stabilizing controllers of a fixed (presumably low) order. Suppose a plant $S = b/a$ is given and suppose that we have a stabilizing controller $\bar{R} = q/p$. We seek to find a stabilizing controller $R = y/x$ of a given order m, if such a controller exists.

The two stabilizing controllers are related as $p = x + bW$, $q = y - aW$, where $W = w/d$. Then

$$\underbrace{\begin{bmatrix} d & 0 & -p & b \\ 0 & d & -q & -a \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \\ d \\ w \end{bmatrix} = 0. \tag{3.51}$$

Let

$$\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \\ d_1 & d_2 \\ w_1 & w_2 \end{bmatrix}$$

be a minimal polynomial basis of A. Then all the stabilizing controllers for S are generated by the formula

$$R = \frac{\lambda_1 y_1 + \lambda_2 y_2}{\lambda_1 x_1 + \lambda_2 x_2}, \tag{3.52}$$

where λ_1 and λ_2 are polynomials such that $\lambda_1 d_1 + \lambda_2 d_2$ is a stable polynomial. A stabilizing controller of order m exists if

$$\deg \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = m. \tag{3.53}$$

Using the convex inner approximation of the set of stable polynomials, we can optimize over polynomials λ_1 and λ_2 to enforce low degrees of x and y (linear algebraic constraint) as well as stability of d (linear matrix inequality constraint).

The catch is that this parametrization is based on a sufficient, hence potentially conservative, stability condition and that the conservativeness depends on the choice of the central polynomial.

Example 3.11

Consider the plant

$$S(s) = \frac{1}{s(s^2 + s + 10)}$$

of order 3. A stabilizing controller of order 2 can be found by placing the closed-loop poles at arbitrary locations. For example, the controller

$$\bar{R}(s) = \frac{-26s^2 + 45s + 1}{s^2 + 4s - 4}$$

places all five closed-loop poles at -1.

Our aim is to find a stabilizing controller of a lower order. A minimal polynomial basis for the polynomial matrix A is given by

$$\begin{bmatrix} 0 & 1 \\ -1 & -26 \\ -1 & s^3+s^2+10s-26 \\ s^2+4s-4 & 149s-103 \end{bmatrix}$$

All the stabilizing controllers can be recovered from the polynomials λ_1 and λ_2 such that the pole polynomial $d=-\lambda_1+\lambda_2(s^3+s^2+10s-26)$ is stable.

From the first two rows of the basis a controller of order 0 can be obtained by restricting the parameters λ_1 and λ_2 to be constant. Hurwitz stability criterion then reveals that d is stable if and only if $\lambda_1 \in (-36,-26)$ and $\lambda_2 = 1$. For example, with $\lambda_1 = -30$ we obtain the controller $R(s)=4$ and the closed-loop pole polynomial $d = s^3+s^2+10s+4$.

In this simple example, we were able to obtain an exact solution. In general, the linear matrix inequality has to be used.

3.6. Concluding Remarks

The parametrization of all stabilizing controllers can easily be extended to *multi-input multi-output* plants. Rational matrices are represented as „polynomial matrix fractions“, that is to say, as the left and right factorizations

$$S = B_R A_R^{-1} = A_L^{-1} B_L \tag{3.54}$$

of two polynomial matrices, where A_R and B_R are right coprime while A_L and B_L are left coprime. The set of all stabilizing controllers for S is given by

$$R = (Y_R - A_R W)(X_R + B_R W)^{-1} = (X_L + W B_L)^{-1}(Y_L - W A_L), \tag{3.55}$$

where the polynomial matrices X_L, Y_L and X_R, Y_R satisfy the Bézout identity

$$\begin{bmatrix} A_L & -B_L \\ Y_L & X_L \end{bmatrix} \begin{bmatrix} X_R & B_R \\ -Y_R & A_R \end{bmatrix} = I, \tag{3.56}$$

and where W is a stable real rational matrix parameter such that $X_L - WB_L$ and $X_R + B_R W$ are nonsingular matrices [22], [23], [36], [39].

It is interesting to note that the set of stabilizing controllers can be parametrized also for plants with *irrational* transfer functions. This is possible whenever such a transfer function is expressed in the form of a fraction of two *coprime* stable functions. This property is by no means evident [36] and it holds, for instance, for transfer functions having a finite number of singularities in $\mathrm{Re}\, s \geq 0$, each of which is a pole.

Even more striking is the observation that stabilizing controllers can be parametrized for *nonlinear* plants, where transfer functions no longer exist. The key condition is again the possibility of factorizing the nonlinear mapping that defines the plant into two „coprime“ mappings, one of them representing a stable system while the other one representing the inverse of a stable system [6]. Technical assumptions may prevent one from parametrizing the *entire* set of internally stabilizing controllers; still, the subset may be large enough for practical purposes.

The parametrization of all stabilizing controllers is a result that launched an entire new area of research and that has ultimately become a new paradigm for the design of linear control systems. Being of algebraic nature [6], [25], it is a result of high generality and elegance. The stabilizing controllers are obtained by solving a linear equation. This is not because the plant to be stabilized is linear but because it is expressed as an element of the *ring of fractions* defined over the ring of stable plants [36]. The requirement of stability is thus expressed as one of divisibility in a ring.

The ring of stable plants depends of course on the notion of stability that is applied. Asymptotically stable systems give rise to transfer functions that are analytic in $\mathrm{Re}\, s \geq 0$, whereas bounded-input bounded-output systems have transfer functions that are analytic in $\mathrm{Re}\, s \geq 0$ as well as at $s = \infty$. That is why we could work with polynomial fractions in this paper; had we required the control system to be bounded-input bounded-output stable, proper stable fractions would be appropriate. The Bézout equation, though solved in a different ring in each case, stands as the fundamental linear design equation.

There is a dual result: the parametrization of all plants that can be stabilized by a fixed controller. This result is useful in system identification. In fact, the (difficult) problem of closed-loop identification of the plant becomes a (simple) problem of open-loop identification of the parameter, as discussed in [1]. Consequently, the parametrization may facilitate the study of dual control.

4. Parametrizace všech stabilizujících regulátorů - State Space Representation of All Stabilizing Controllers

Vladimír Kučera

Stabilita je základním požadavkem při návrhu regulačních systémů. Proto je důležité znát všechny regulátory, které stabilizují danou soustavu. Takových regulátorů je nekonečně mnoho a parametrizace je vhodným způsobem jejich vyjádření. Splnění dalších požadavků na regulační systém pak lze zajistit vhodným výběrem parametru.

Most control systems are designed to be stable and to meet additional specifications, such as optimality and robustness. It is therefore natural to design the systems step by step: stabilization first, then the additional specifications each at a time. For this it is obviously necessary to have any and all solutions of the current step available before proceeding any further.

This motivates the need for all controllers that stabilize a given system. In fact, this is an infinite family and we find it convenient to describe it in a parametric form, known as the Youla-Kučera parameterization. The additional specifications are then met by selecting an appropriate parameter. Such a procedure is simple, systematic, and transparent.

The lecture will start with a transfer function approach to the parameterization of all stabilizing controllers and proceed with a state space approach. It will be shown how doubly coprime fractional representations of a system can be obtained by applying to it a stabilizing state feedback and a stabilizing output injection. Consequently, all controllers that stabilize a given system are built around an observer-based central stabilizing controller.

The lecture has a significant pedagogic value. State space and transfer function techniques are presented as connected approaches, rather than isolated alternatives.

5. Kvadraticky optimální systémy s předepsanými póly - Optimal Control Systems with Prescribed Eigenvalues

Jiří Cigler

Póly lineárního systému lze umístit do požadované polohy buď přímo, metodou přiřazení pólů, nebo nepřímo, například pomocí kvadratické optimalizace. Optimální řízení vyústí v určité rozložení pólů podle zadaného kritéria kvality regulace. Ukážeme, jak kvadratickým kritériem dosáhnout předepsaného rozložení pólů.

5.1. Introduction

The optimal linear-quadratic design has several nice features. In particular, the closed-loop system enjoys certain robustness properties. The transient behavior of the closed-loop system, however, is difficult to determine in advance since there is no simple relation between the weighting matrices that specify the performance index and the closed-loop eigenvalues. To get a good transient response, the weights are often determined iteratively through trial and error.

The eigenvalue assignment (or pole placement) methods address the transient phenomena directly by specifying a set of desired closed-loop eigenvalues. However, different feedback gains can lead to the same pole pattern when the system has several inputs and these gains can produce different transients.

Attempts to combine the two methods are of an early date. Results exist on optimal control with eigenvalues restricted to a specified region of the complex plane, namely a semi-plane [2], a disk [9], a sector [11], or a hyperbolic region [19]. Optimal control with exactly prescribed eigenvalues is more difficult to achieve. Various results reflect various approaches to seeking a relationship between the weighting matrices and eigenvalue locations [33], [35], [8], [20], [4].

This paper is a streamlined presentation of paper [4], with some generalizations. The weighting matrices of the optimal control problem are constructed so as to relocate a single eigenvalue (or a pair of complex conjugate eigenvalues) to a prescribed position while leaving the remaining eigenvalues at their original positions. The region of the complex plane into which each eigenvalue can be located is described. The process of relocation can be repeated as long as a desired eigenvalue pattern is achieved.

5.2. Preliminaries

We consider a continuous-time linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad t \geq 0 \tag{5.1}$$

and seek a control law

$$u(t) = Fx(t)$$

that minimizes a quadratic cost of the form

$$\int_0^\infty (x^T Qx + u^T Ru)dt \tag{5.2}$$

for every initial state $x(0)$. The matrices Q and R are symmetric with $Q = C^T C \geq 0$ and $R > 0$.

We suppose that the pair (A, B) is stabilizable and the pair (A, C) is detectable. Then

$$F = -R^{-1}B^T P, \tag{5.3}$$

where P is a unique symmetric solution of the algebraic Riccati equation

$$PA + A^T P - PBR^{-1}B^T P + Q = 0 \tag{5.4}$$

such that $P \geq 0$. The optimal closed-loop system

$$\dot{x}(t) = (A + BF)x(t) \tag{5.5}$$

is asymptotically stable.

Consider the Hamiltonian matrix

$$H = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \tag{5.6}$$

The eigenvalues of H are symmetrically distributed with respect to the imaginary axis. One half of the eigenvalues have negative real parts; they are the eigenvalues of the optimal closed-loop system (5.5).

5.3. Single Eigenvalue Relocation

The linear-quadratic control imposes the eigenvalues of the closed-loop system. Given A and B, the choice of Q and R achieves a certain pattern of the eigenvalues of H, which in turn define the closed-loop system eigenvalues.

In order to achieve an optimal system with prescribed eigenvalues, we shall investigate the possibility of selecting Q and R so as to relocate (or shift) a single eigenvalue at a time, leaving the remaining eigenvalues at their original positions. For the sake of exposition, we shall consider the two cases as follows.

5.3.1. *The Case of a Real Simple Eigenvalue*

Let T be a similarity transformation that brings A to its Jordan form,

$$\tilde{A} = T^{-1}AT, \quad \tilde{B} = T^{-1}B \tag{5.7}$$

and suppose that $\tilde{A}$ is diagonal.

Choose one controllable eigenvalue, say λ_1, of A to be re-located and exhibit it in the Jordan form as follows

$$\tilde{A} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & J_1 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} b_1^T \\ \times \end{bmatrix} \tag{5.8}$$

where b_1^T is the first row of matrix $\tilde{B}$ and $\times$ indicates the remaining entries.

Take the weighting matrix Q as

$$Q = (T^{-1})^T \tilde{Q} T^{-1}, \tag{5.9}$$

where

$$\tilde{Q} = \begin{bmatrix} q_1 & 0 \\ 0 & 0 \end{bmatrix} \tag{5.10}$$

with $q_1 \geq 0$ a real parameter, and select the weighting matrix R so that $b_1^T R^{-1} b_1 = 1$.

Let μ_1 be the desired position to which the eigenvalue λ_1 is to be relocated. We shall first analyze which positions for μ_1 are admissible and which matrices Q will realize the shift.

Consider the Hamiltonian matrix (5.6),

$$H = \begin{bmatrix} T & 0 \\ 0 & (T^{-1})^T \end{bmatrix} \begin{bmatrix} \tilde{A} & -\tilde{B}R^{-1}\tilde{B}^T \\ -\tilde{Q} & -\tilde{A}^T \end{bmatrix} \begin{bmatrix} T^{-1} & 0 \\ 0 & T^T \end{bmatrix},$$

and calculate

$$\begin{aligned} \det(sI - H) &= \det \begin{bmatrix} sI - \tilde{A} & \tilde{B}R^{-1}\tilde{B}^T \\ \tilde{Q} & sI + \tilde{A}^T \end{bmatrix} \\ &= \det \begin{bmatrix} s - \lambda_1 & 0 & 1 & \times \\ 0 & sI - J_1 & \times & \times \\ q_1 & 0 & s + \lambda_1 & 0 \\ 0 & 0 & 0 & s + J_1^T \end{bmatrix} \\ &= \det(sI - J_1)\det(sI + J_1^T)\det(sI - H_1), \end{aligned}$$

where

$$H_1 = \begin{bmatrix} \lambda_1 & -1 \\ -q_1 & -\lambda_1 \end{bmatrix}$$

and where $\times$ indicates the remaining entries. It follows that all the eigenvalues of A but λ_1 remain unchanged and the re-location of λ_1 to μ_1 requires that

$$\det(sI - H_1) = (s - \mu)(s + \mu),$$

that is,

$$\det(sI - H_1) = s^2 - (\lambda_1^2 + q_1) = s^2 - \mu_1^2 .$$

We conclude that $|\mu_1| \geq |\lambda_1|$ since $q_1 \geq 0$. The eigenvalue can only be relocated further from the origin. In particular, if μ_1 is to be stable, it can only be shifted to the left.

Having chosen Q and R, the optimal control law (5.3) that achieves the desired shift is given by solving the Riccati equation (5.4). Make an inspired guess that the optimal solution matrix P is

$$P = (T^{-1})^T \tilde{P} T^{-1}, \tag{5.11}$$

where

$$\tilde{P} = \begin{bmatrix} p_1 & 0 \\ 0 & 0 \end{bmatrix} \tag{5.12}$$

for some real constant $p_1 \geq 0$. Substituting (5.7), (5.9) and (5.11) into (5.4) gives

$$(T^{-1})^T (\tilde{P}\tilde{A} + \tilde{A}^T \tilde{P} - \tilde{P}\tilde{B}\tilde{R}^{-1}\tilde{B}^T \tilde{P} + \tilde{Q}) T^{-1} = 0 .$$

Using (5.8), (5.10) and (5.12), the Riccati equation is reduced to a scalar equation for p_1, namely

$$p_1^2 - 2\lambda_1 p_1 - q_1 = 0,$$

which can readily be solved. In particular, $p_1 = \lambda_1 - \mu_1$.

The process can be repeated for each eigenvalue *ad libitum* until the desired pattern of eigenvalues is achieved.

5.3.2. *The Case of a Real Multiple Eigenvalue*

Now suppose that the controllable eigenvalue of A to be relocated, call it again λ_1, is real but it generates a Jordan block of size k.

We claim that the result obtained in Subsection A holds in this case also. Indeed, the choice of Q as described above results in a shift of λ_1 to a new position μ_1. The remaining eigenvalues of A keep their original positions. In particular, λ_1 remains an eigenvalue of A but it generates a Jordan block of size $k-1$.

Therefore, the effect of the optimal control law (5.3) on sys-tem (5.1) is to split the Jordan block of λ_1 into a single eigenvalue μ_1 and a smaller block of λ_1. This process can be continued, resulting in a spectrum of k eigenvalues $\mu_1, \mu_2, \ldots, \mu_k$ positioned outward of λ_1.

5.4. Relocation of a Complex Conjugate Pair of Eigenvalues

Suppose that A has a pair of simple, complex conjugate eigenvalues, say $\lambda_1 = \lambda$ and $\lambda_2 = \bar{\lambda}$, which are controllable and are to be relocated simultaneously to obtain a new complex conjugate pair of eigenvalues $\mu_1 = \mu$ and $\mu_2 = \bar{\mu}$. In this case we have, using an appropriate similarity transformation T,

$$\tilde{A} = \begin{bmatrix} \Lambda_2 & 0 \\ 0 & J_2 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B_2^T \\ \times \end{bmatrix} \tag{5.13}$$

where

$$\Lambda_2 = \begin{bmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{bmatrix} \tag{5.14}$$

and where B_2^T denotes the first two rows of $\tilde{B}$ and $\times$ indicates the remaining entries.

Take the weighting matrix Q as

$$Q = (\bar{T}^{-1})^T \tilde{Q} T^{-1}, \tag{5.15}$$

where

$$\tilde{Q} = \begin{bmatrix} Q_2 & 0 \\ 0 & 0 \end{bmatrix} \tag{5.16}$$

and $Q_2 \geq 0$ is a Hermitian 2×2 matrix parameter. As the first two columns of T are complex conjugate of each other, Q_2 will have equal diagonal entries,

$$Q_2 = \begin{bmatrix} q & q_{12} \\ \bar{q}_{12} & q \end{bmatrix} \tag{5.17}$$

for a real q and a complex q_{12} that satisfy $q \geq |q_{12}|$. Select the weighting matrix R so that

$$\bar{B}_2^T R^{-1} B_2 = \begin{bmatrix} 1 & \bar{\omega} \\ \omega & 1 \end{bmatrix} := \Omega_2 \tag{5.18}$$

for a complex ω such that $|\omega| \leq 1$.

For a single eigenvalue, only an outward shift is possible. The situation is more involved in the case of shifting a pair of eigenvalues. The relevant quantities are related by the 4×4 Hamiltonian matrix

$$H_2 = \begin{bmatrix} \Lambda_2 & -\Omega_2 \\ -Q_2 & -\overline{\Lambda}_2^T \end{bmatrix}$$

whose eigenvalues are to equal $\mu, \overline{\mu}$ and $-\mu, -\overline{\mu}$. Substituting from (5.14), (5.17) and (5.18), we obtain

$$\det(sI - H_2) = \det \begin{bmatrix} s-\lambda & 0 & 1 & \overline{\omega} \\ 0 & s-\overline{\lambda} & \omega & 1 \\ q & q_{12} & s+\overline{\lambda} & 0 \\ \overline{q}_{12} & q & 0 & s+\lambda \end{bmatrix}$$

$$= s^4 - 2(\operatorname{Re}\lambda^2 + q + \operatorname{Re}\omega q_{12})s^2 + $$
$$|\lambda|^4 + 2|\lambda|^2 q + 2\operatorname{Re}\lambda^2 \omega q_{12} + (1-|\omega|^2)(q^2 - |q_{12}|^2).$$

The intended relocation calls for

$$\det(sI - H_2) = s^4 - 2\operatorname{Re}\mu^2 s^2 + |\mu|^4$$

and the admissible region into which $\lambda,\ \overline{\lambda}$ can be relocated is determined by the equalities

$$\operatorname{Re}\mu^2 = \operatorname{Re}\lambda^2 + q + \operatorname{Re}\omega q_{12} \tag{5.19}$$

$$|\mu|^4 = |\lambda|^4 + 2|\lambda|^2 q + 2\operatorname{Re}\lambda^2 \omega q_{12} + (1-|\omega|^2)(q^2 - |q_{12}|^2)\,. \tag{5.20}$$

The shape of the admissible region for $\mu, \overline{\mu}$ depends on λ and ω. To visualize the region, we denote $x = \operatorname{Re}\mu$, $y = \operatorname{Im}\mu$ so as to have

$$\operatorname{Re}\mu^2 = x^2 - y^2, \quad |\mu|^4 = (x^2 + y^2)^2$$

and we proceed by fixing the values of ω as follows.

5.4.1. *The Case of* $|\omega| = 1$

In this case Ω is a rank-one singular matrix, which happens for single-input systems. Equations (5.19) and (5.20) read

$$x^2 - y^2 = \operatorname{Re}\lambda^2 + q + \operatorname{Re}\omega q_{12} \tag{5.21}$$

$$(x^2 + y^2)^2 = |\lambda|^4 + 2|\lambda|^2 q + 2\operatorname{Re}\lambda^2 \omega q_{12} \tag{5.22}$$

We observe that these equations are linear in q and are to be solved for some real $q \geq |q_{12}|$.

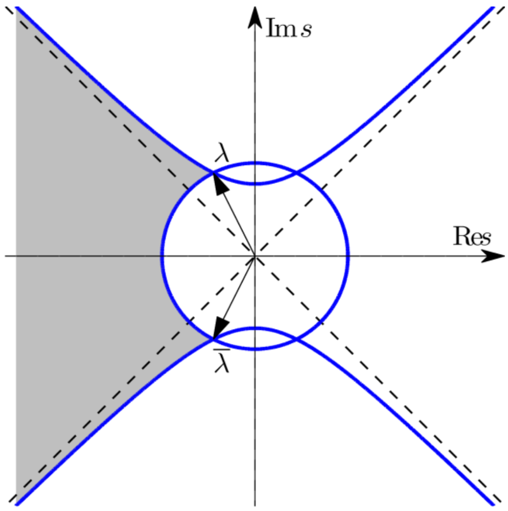

Figure 5.1 Stable admissible region for *λ* = – 1 + 2*j* and for |*ω*| = 1.

Therefore suppose that $q \geq |q_{12}|$. Then

$$|\operatorname{Re}\omega q_{12}| \leq |\omega q_{12}| \leq |q_{12}| \leq q$$

and

$$|\operatorname{Re}\lambda^2 \omega q_{12}| \leq |\lambda^2 \omega q_{12}| \leq |\lambda^2||q_{12}| \leq |\lambda^2| q.$$

In view of that,

$$q + \operatorname{Re}\omega q_{12} \geq 0, \quad 2|\lambda|^2 q + 2\operatorname{Re}\lambda^2 \omega q_{12} \geq 0$$

and (5.21), (5.22) yield the inequalities

$$x^2 - y^2 \geq \operatorname{Re}\lambda^2 \tag{5.23}$$

$$x^2 + y^2 \geq |\lambda|^2. \tag{5.24}$$

Observe that (5.23) represents either the interior of the equi-lateral hyperbola

$$x^2 - y^2 \geq \mathrm{Re}\,\lambda^2,$$

or the exterior of the conjugated hyperbola

$$y^2 - x^2 \leq -\mathrm{Re}\,\lambda^2,$$

or the sector delineated by their asymptotes

$$y = x, \quad y = -x,$$

depending on the sign of $\mathrm{Re}\lambda^2$. The real and imaginary axes of the above hyperbolas equal the square root of $|\mathrm{Re}\lambda^2|$.

Inequality (5.24) represents the exterior of a circle with radius $|\lambda|$, centered at the origin.

Figure 5.1 and Figure 5.2 visualize as shaded areas the stable admissible regions into which the eigenvalues $\lambda = -1 + 2j$, $\lambda = 3 + 2j$ can be relocated when $|\omega| = 1$.

5.4.2. *The Case of $\omega = 0$*

In this case Ω is the identity matrix. Equations (5.19) and (5.20) read

$$x^2 - y^2 = \mathrm{Re}\,\lambda^2 + q \tag{5.25}$$

$$(x^2 + y^2)^2 = |\lambda|^4 + 2|\lambda|^2 q + q^2 - |q_{12}|^2. \tag{5.26}$$

We observe that these equations are quadratic in q and are to be solved for some real q and a complex q_{12} such that $q \geq |q_{12}|$. It follows from (5.25) that q is real as long as (5.26) is satisfied for some q_{12}.

Write (5.26) in the form

$$|q_{12}|^2 = (q + |\lambda|^2)^2 - (x^2 + y^2)^2.$$

In view of $|q_{12}| \geq 0$ this equation implies the inequality

$$x^2 + y^2 \leq q + |\lambda|^2.$$

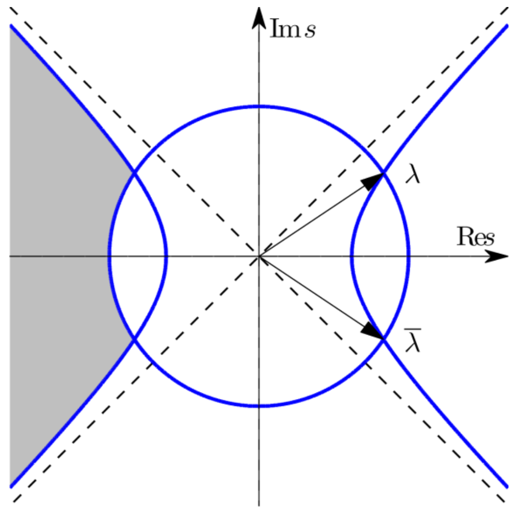

Figure 5.2 Stable admissible region for λ = 3 + 2j and for |ω| = 1.

Substituting for q from (5.25), one obtains

$$2y^2 \leq |\lambda|^2 - \mathrm{Re}\,\lambda^2$$

or equivalently

$$y^2 \leq \mathrm{Im}^2\lambda \tag{5.27}$$

On the other hand, the condition $q \geq |q_{12}|$ turns (5.26) into the inequality

$$(x^2 + y^2)^2 \geq |\lambda|^4 + 2|\lambda|^2 + q^2.$$

Substituting for q from (5.25), one obtains

$$(x^2 + y^2)^2 - 2|\lambda|^2(x^2 - y^2) + |\lambda|^4 \geq 4|\lambda|^2\,\mathrm{Im}^2\lambda \tag{5.28}$$

Observe that equation (5.27) defines a strip of width $2|\mathrm{Im}\lambda|$ along the real axis while equation (5.28) represents the exterior of a Cassini oval with foci at the points $(x, y) = (|\lambda|, 0)$ and $(x, y) = (-|\lambda|, 0)$. The shape of the Cassini oval depends on the value of $4(\mathrm{Im}^2\lambda)/|\lambda|^2$. Thus the real part of the eigenvalues $\lambda, \bar{\lambda}$ can only be relocated outside the oval while the imaginary part is not increased.

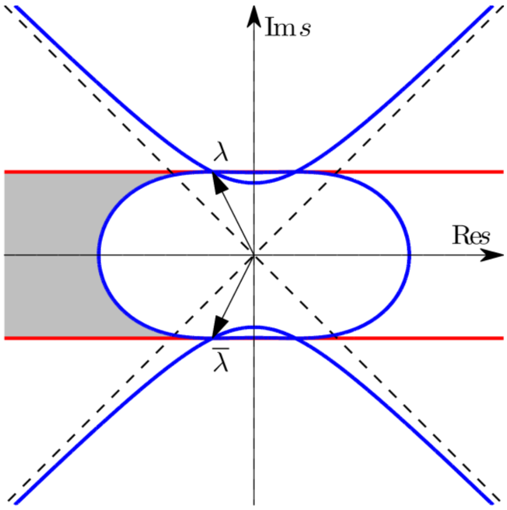

Figure 5.3 Stable admissible region for λ = – 1 + 2*j* and for ω = 0

Figure 5.3 and Figure 5.4 visualize the stable admissible regions – the shaded areas – into which the eigenvalues $\lambda = -1 + 2j$, $\lambda = 3 + 2j$ can be relocated when $\omega = 0$. The ovals are shown in blue whereas the strip boundaries are shown in red.

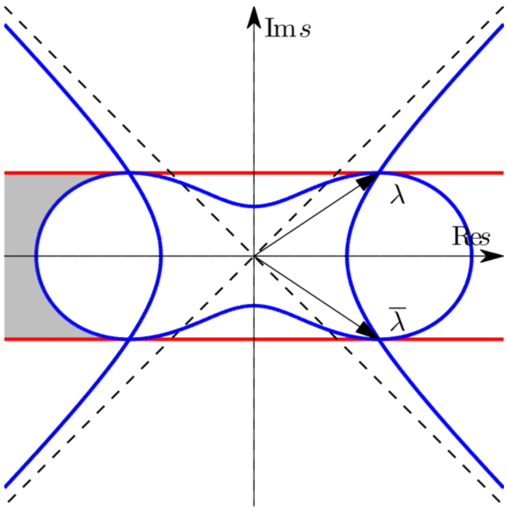

Figure 5.4 Stable admissible region for λ = 3 + 2*j* and for ω = 0.

5.4.3. *The Case of* $0 < |\omega| < 1$

In this case Ω is a general rank-two matrix. The shape of the admissible region can be investigated using (5.19) and (5.20) while considering the conditions for a real q and a complex q_{12} to exist such that $q \geq |q_{12}|$.

Equations (5.19) and (5.20) read

$$x^2 - y^2 = \operatorname{Re}\lambda^2 + q + \operatorname{Re}\omega q_{12} \tag{5.29}$$

$$(x^2 + y^2)^2 = |\lambda|^4 + 2|\lambda|^2 q + 2\operatorname{Re}\lambda^2\omega q_{12} + (1-|\omega|^2)(q^2 - |q_{12}|^2). \tag{5.30}$$

It follows from (5.29) that q is real as long as (5.30) is satisfied for some q_{12}. The condition $q \geq |q_{12}|$ turns (5.30) into the in-equality

$$(x^2 + y^2)^2 \geq |\lambda|^4 + 2|\lambda|^2 q + 2\operatorname{Re}\lambda^2\omega q_{12}. \tag{5.31}$$

Now (5.29) and (5.31) jointly define regions bounded by a family of octic curves parameterized by ω. The curves are shown for the eigenvalue $\lambda = -1 + 2j$ in Figure 5.5 and Figure 5.6 in blue, each figure corresponding to a particular value of ω.

On the other hand, the upper bound for the imaginary part y of μ is evaluated from (5.29) and (5.30) for each x. The result is a family of curves parameterized by ω. The curves are shown for the eigenvalue $\lambda = -1 + 2j$ in Figure 5.5 and Figure 5.6 in red, each figure corresponding to a particular value of ω.

The shaded areas that are shown in Figure 5.5 and Figure 5.6 por-tray the regions into which $\mu, \bar{\mu}$ can be relocated. Thus the real part of the eigenvalues can be shifted outward while the imaginary part is bounded from above.

Note that when $\omega \to 0$, the attainable regions shown in Figure 5.5 and Figure 5.6 approach the region shown in Figure 5.3. On the other hand, when $|\omega| \to 1$, we recover the singular case, see Figure 4.1. It is of interest to note that the maximal assignable imaginary part in Figure 5.5 and Figure 5.6 grows progressively with ω. The growth is slow for $\omega < 0.5$ and is fast only when $\omega > 0.9$.

The target eigenvalues can in particular be taken real, resulting in a double real eigenvalue μ. This case is addressed by setting $y = \operatorname{Im}\mu = 0$ in the expressions above.

Having chosen Q and R, the optimal control law (5.3) that achieves the desired shift is obtained by solving the Riccati equation (5.4). Make an inspired guess that the optimal solution matrix P is

$$P = (T^{-1})^T \tilde{P} T^{-1}, \tag{5.32}$$

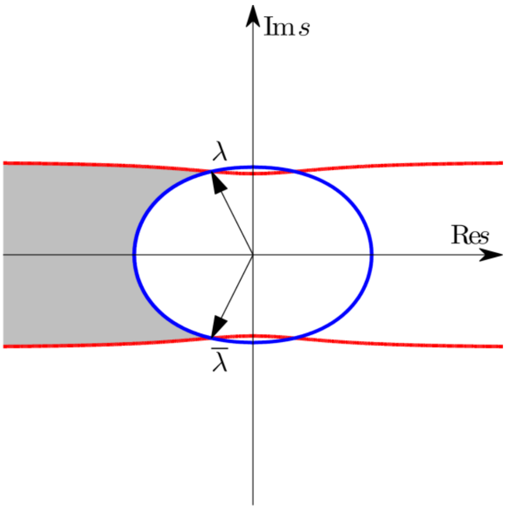

Figure 5.5 Stable admissible region for $\lambda = -1 + 2j$ and for $|\omega| = 0.5$.

where

$$\tilde{P} = \begin{bmatrix} P_2 & 0 \\ 0 & 0 \end{bmatrix} \tag{5.33}$$

for some 2×2 Hermitian matrix $P_2 \geq 0$ having equal diagonal entries. Substituting (5.7), (5.15) and (5.32) into (5.4) yields

$$(T^{-1})^T (\tilde{P}\tilde{A} + \tilde{A}^T \tilde{P} - \tilde{P}\tilde{B}\tilde{R}^{-1}\tilde{B}^T \tilde{P} + \tilde{Q}) T^{-1} = 0.$$

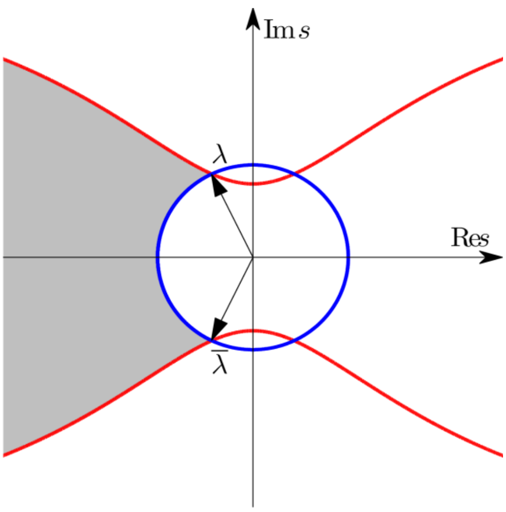

Figure 5.6 Stable admissible region for $\lambda = -1 + 2j$ and for $|\omega| = 0.95$.

Using (5.13), (5.16), (5.18) and (5.33), the Riccati equation reduces to

$$P_2\Lambda_2 + \overline{\Lambda}_2^T P_2 - P_2\Omega_2 P_2 + Q_2 = 0,$$

to be solved for P_2.

The process can be repeated for each pair of complex conjugate eigenvalues *ad libitum* until the desired eigenvalue pattern is achieved.

5.5. Conclusion

A design of linear-quadratic optimal systems with prescribed eigenvalues has been presented. The method is well suited to modify a given linear-quadratic design so as to improve the transient response of the closed-loop system. Slow eigenvalues can be made faster and oscillatory eigenvalues can be dampened. The didactic value of the results can be seen in providing an explicit relationship between the weighting matrices and the closed-loop eigenvalue positions. The method is so simple that it can eventually make its way to control textbooks.

6. JE KONEČNÝ POČET KROKŮ REGULACE KVADRATICKY OPTIMÁLNÍ? - DEADBEAT RESPONSE IS L_2 OPTIMAL

Vladimír Kučera

Konečný počet kroků regulace je specifickým požadavkem při návrhu diskrétně pracujících regulačních obvodů. Naproti tomu kvadratické kritérium je uznávanou metodou návrhu stabilních a optimálních systémů. Je překvapující, že systémy zaručující konečný počet kroků regulace jsou kvadraticky optimální.

6.1. Deadbeat Regulator

We consider a linear system (A, B) described by the state equation

$$x_{k+1} = Ax_k + Bu_k, \quad k = 0, 1, \ldots \tag{6.1}$$

where $u_k \in I\!R^m$ and $x_k \in I\!R^n$. The objective of *deadbeat regulation* is to determine a linear state feedback of the form

$$u_k = -Lx_k \tag{6.2}$$

that drives each initial state x_0 to the origin in a least number of steps.

We define the *reachability subspaces* of system (6.1) by

$$R_0 = 0,$$
$$R_k = \text{range}[B \;\; AB \;\; \ldots \;\; A^{k-1}B], \quad k = 1, 2, \ldots .$$

Hence R_k is the set of states of (6.1) that can be reached from the origin in k steps by applying an input sequence $u_0, u_1, \cdots, u_{k-1}$. When $R_n = I\!R^n$, the system (A, B) of (6.1) is said to be *reachable*.

Define the integers

$$q_k = \text{dimension } R_k - \text{dimension } R_{k-1}$$

and for $k = 1,2,\cdots, m$ let

$$r_i = \text{cardinality}\,\{q_k : q_k \geq i\}.$$

The integers $r_1 \geq r_2 \geq \cdots \geq r_m$ are the *reachability indices* of system (6.1).

We further define the controllability subspaces for (6.1) by

$$C_0 = 0,$$
$$C_k = \{x \in I\!R^n : F^k x \in R_k\}, \quad k = 1, 2, \ldots$$

Thus C_k is the set of all states of (6.1) that can be steered to the origin in k steps by an appropriate control sequence $u_0, u_1, \cdots, u_{k-1}$. When $C_n = I\!R^n$, the system (A, B) of (6.1) is said to be *controllable*. It follows from the definitions that reachability implies controllability and the converse is true whenever A is nonsingular.

The existence and construction of deadbeat control laws is described below. For each $k = 1, 2, \cdots$ let $S_1, S_2, \cdots, S_k$ be a sequence of $m \times q_1, m \times q_2, \ldots, m \times q_k$ matrices such that

$$\text{range}\,[BS_1 \;\; ABS_2 \;\; \ldots \;\; A^{k-1}BS_k] = \text{range}\, R_k.$$

Therefore $S_1, S_2, \cdots, S_k$ serve to select a basis for R_k.

Theorem 6.1

[29] There exists a deadbeat control law (6.2) if and only if the system (A, B) of (6.1) is controllable. Let

$$L_0 = 0,$$
$$L_k = L_{k-1} + L'_k (A - BL_{k-1})^k, \quad k = 1, 2, \ldots \tag{6.3}$$

where L'_k satisfies

$$L'_k [BS_1 \;\; ABS_2 \;\; \ldots \;\; A^{k-1}BS_k] = [0 \;\; \ldots \;\; 0 \;\; S_k].$$

Then $L = L_n$ is a deadbeat regulator gain.

The theorem identifies all deadbeat control laws via the recursive procedure (6.3). Actually the procedure can be terminated in q steps, where $q = \min\,\{k : C_{k+1} = C_k\}$. The resulting closed-loop system matrix is nilpotent with index q,

$$(A - BL)^q = 0. \tag{6.4}$$

If A is nonsingular, the recursive procedure (6.3) can be shortcut by setting $L_{q-1} = 0$. Then, the Jordan structure of $A - BL$ comprises m nilpotent blocks [28] of sizes $r_1, r_2, \cdots, r_m$ and the index of nilpotency equals $q = r_1$. In fact, this is the least size of Jordan blocks that can be achieved [24] in a reachable system (6.1) by applying state feedback (6.2).

6.2. Linear Quadratic Regulator

We consider a linear system described by the state equation (6.1),

$$x_{k+1} = Ax_k + Bu_k, \quad k = 0, 1, \ldots$$

where $u_k \in I\!R^m$ and $x_k \in I\!R^n$. The objective of LQ regulation is to find a linear state feedback of the form (6.2),

$$u_k = -Lx_k$$

that stabilizes the closed-loop system

$$x_{k+1} = (A - BL)x_k$$

and, for every initial state x_0, minimizes the l_2 norm

$$\|y\|^2 = \sum_{k=0}^{\infty} y_k^{\mathrm{T}} y_k$$

of a specified output $y_k \in I\!R^p$ of the form

$$y_k = Cx_k + Du_k. \tag{6.5}$$

The existence and construction of an LQ control law is described below. We say that the system (A, B) of (6.1) is *stabilizable* if the system matrices can be transformed to the following form using an appropriate basis:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$$

where the subsystem defined by the pair of matrices (A_{11}, B_1) is reachable and A_{22} is a stable matrix. We say that the system (A, B, C, D) defined by the state equation (6.1) and the output equation (6.5) is *left invertible* if its transfer function $C(zI_n - A)^{-1}B + D$ has full column normal rank. We further define the *system matrix* as the polynomial matrix

$$S(z) = \begin{bmatrix} zI_n - A & -B \\ C & D \end{bmatrix}$$

and say that a complex number ξ is an *invariant zero* of the system (A, B, C, D) if the rank of $S(\xi)$ is strictly less than the normal rank of $S(z)$.

Theorem 6.2

[32] Suppose that the system (A, B) of (6.1) is stabilizable. Suppose that the system (A, B, C, D) defined by (6.1) and (6.5) is left invertible and also has no invariant zeros on the unit circle $|z| = 1$. Then, there exists a unique LQ regulator gain given by

$$L = (D^T D + B^T XB)^{-1}(B^T XA + D^T C), \tag{6.6}$$

where X is the largest symmetric nonnegative definite solution of the algebraic Riccati equation

$$\begin{aligned} X &= A^T XA + C^T C \\ &\quad -(B^T XA + D^T C)^T (D^T D + B^T XB)^{-1}(B^T XA + D^T C). \end{aligned} \tag{6.7}$$

The assumption of left invertibility for (A, B, C, D) is required in order for the matrix $D^T D + B^T XB$ to be nonsingular while the remaining assumptions are required for the existence of the requisite solution X of the algebraic Riccati equation.

6.3. The Deadbeat Regulator As an LQ Regulator

The aim of this section is to show that deadbeat control laws in *reachable* systems are LQ optimal. This will be done by constructing an output (6.5) of the system (6.1) so that the resulting LQ regulator gain is a deadbeat gain.

Let the system (A, B) of (6.1) be reachable with reachability indices $r_1, r_2, \cdots, r_m$. Then there exists a similarity transformation T that brings the matrices A and B to the standard reachability form [28],

$$A' = TAT^{-1}, \quad B' = TB \tag{6.8}$$

where A' is a top-companion matrix with nonzero entries in rows r_i, $i = 1,2,\ldots,m$ and B' has nonzero entries only in rows r_i and columns $j \geq i$, $i = 1,2,\ldots,m$.

Theorem 6.3

Suppose that the system (A, B) of (6.1) is reachable, with reachability indices $r_1 \geq r_2 \geq \cdots \geq r_m$ and with the matrix B having rank m. Let T be a similarity transformation that brings A and B to the standard reachability form. Then, the feedback gain L that is LQ optimal with respect to $C = T$ and $D = 0$ in (6.5) is a deadbeat gain.

Proof. Consider the transfer function of system (6.1) in polynomial matrix fraction form

$$(zI_n - A)^{-1}B = Q(z)P^{-1}(z) \tag{6.9}$$

where P and Q are right coprime polynomial matrices in z of respective size $m \times m$ and $n \times m$, with P column reduced and column-degree ordered with column degrees $r_1 \geq r_2 \geq \cdots \geq r_m$. These integers are the reachability indices of (6.1).

The system (6.1) being reachable, the matrices $zI_n - A$ and B are left coprime. It follows from (6.9) that the denominator matrices $zI_n - A$ and $P(z)$ have the same determinant (in fact, the same invariant polynomials).

For any feedback (6.2) applied to system (6.1), one obtains

$$[zI_n - A \quad -B]\begin{bmatrix} I_n & 0 \\ -L & I_m \end{bmatrix} = [zI_n - (A - BL) \quad -B]$$

and

$$\begin{bmatrix} I_n & 0 \\ L & I_m \end{bmatrix}\begin{bmatrix} Q(z) \\ P(z) \end{bmatrix} = \begin{bmatrix} Q(z) \\ P(z) + LQ(z) \end{bmatrix}.$$

Then (6.9) implies that

$$[zI_n - (A - BL)]^{-1}B = Q(z)[P(z) + LQ(z)]^{-1}. \tag{6.10}$$

Thus the closed-loop system transfer function matrices $zI_n - (A - BL)$ and B are left coprime while $P(z) + LQ(z)$ and $Q(z)$ are right coprime. It follows from (6.10) that the polynomial matrices $zI_n - (A - BL)$ and $P(z) + LQ(z)$ have the same determinant (in fact, the same invariant polynomials).

Now we show that an LQ regulator gain exists that is optimal with respect to $C = T$ and $D = 0$. Indeed, the pair (A, B) is reachable, hence stabilizable. The quadruple $(A, B, T, 0)$ corresponds to the transfer function $T(zI_n - A)^{-1}B$ whose column normal rank is m, hence the system is left invertible. The system matrix

$$S(z) = \begin{bmatrix} zI_n - A & -B \\ T & 0 \end{bmatrix}$$

has rank $n + m$ for all complex numbers z, which implies that $(A, B, T, 0)$ has no invariant zeros at all. Consequently, the assumptions of Theorem 6.2 are all satisfied, which shows the existence of an LQ optimal regulator gain (6.6).

Consider the associated algebraic Riccati equation (6.7). Add $z^{-1}(XA - XA) + (XA - XA)^{\mathrm{T}} z$ to the right-hand side of the equation in order to introduce polynomial matrix factorizations, then use (6.6) and (6.9) to get the following identity [28]

$$\begin{aligned} &[P(z^{-1}) + LQ(z^{-1})]^{\mathrm{T}} (D^T D + B^{\mathrm{T}} XB)[P(z) + LQ(z)] \\ &\quad = [CQ(z^{-1}) + DP(z^{-1})]^{\mathrm{T}} [CQ(z) + DP(z)]. \end{aligned} \tag{6.11}$$

Define a polynomial $m \times m$ matrix F, which is column reduced and column-degree ordered with column degrees $r_1 \geq r_2 \geq \cdots \geq r_m$, by the equation

$$F^{\mathrm{T}}(z^{-1})F(z) = [CQ(z^{-1}) + DP(z^{-1})]^{\mathrm{T}} [CQ(z) + DP(z)] \tag{6.12}$$

in such a way that its inverse F^{-1} is analytic in the domain $|z| \geq 1$. This matrix is referred to as the *spectral factor* and it is determined uniquely by (6.12) up to multiplication on the left by a constant orthogonal matrix.

The pair (A, B) being reachable, the matrices A and B can be brought to the reachability standard form (6.8) using the similarity transformation matrix T. The corresponding right coprime polynomial fraction matrices are related by

$$P'(z) = P(z), \quad Q'(z) = TQ(z)$$

and Q' has the block-diagonal form

$$Q'(z) = \text{block-diag}\left[\begin{bmatrix} 1 \\ z \\ \vdots \\ z^{r_1-1} \end{bmatrix}, \begin{bmatrix} 1 \\ z \\ \vdots \\ z^{r_2-1} \end{bmatrix}, \dots, \begin{bmatrix} 1 \\ z \\ \vdots \\ z^{r_m-1} \end{bmatrix} \right].$$

The spectral factorization (6.12) reads

$$\begin{aligned} F^{\mathrm{T}}(z^{-1})F(z) &= Q^{\mathrm{T}}(z^{-1})T^{\mathrm{T}}TQ(z) \\ &= Q'^{T}(z^{-1})Q'(z) = \text{diag}[r_1, r_2, \dots, r_m] \end{aligned}$$

so that

$$F(z) = \text{diag}[\sqrt{r_1}z^{r_1}, \sqrt{r_2}z^{r_2}, \dots, \sqrt{r_m}z^{r_m}].$$

It follows from (6.11) that the LQ regulator that is optimal with respect to $C = T$ and $D = 0$ induces the closed-loop right denominator matrix $P(z) + LQ(z)$ with invariant factors $z^{r_1}, z^{r_2}, \dots,, z^{r_m}$. The same factors are shared by the closed-loop left denominator matrix $zI_n - (A - BL)$. Therefore, $A - BL$ is nilpotent with Jordan structure comprising m nilpotent blocks of sizes $r_1, r_2, \dots r_m$. The nilpotency index of $A - BL$ is r_1, the largest reachability index. Q.E.D.

The restriction of Theorem 6.3 to reachable systems, while technically important, is actually a mild restriction as it covers the case of main practical interest. Controllable systems (6.1) that are not reachable possess a singular matrix A with nilpotent dynamics. Such systems are inherently discrete. In particular, the periodically sampled continuous time systems, considered in the discrete instants of time, have a nonsingular matrix A.

The other restriction applied in Theorem 6.3, namely B having full column rank m, is needed to guarantee the solvability of the LQ regulator. It represents no practical constraint, either. Indeed, if the rank of B is less than m, then the components of the control vector u are linearly dependent.

6.4. Example

To illustrate, let us consider a system (6.1) described by the matrices

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Since

$$R_0 = 0,\ R_1 = \text{image}\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},\ R_2 = R_3 = \text{image}\left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right] = I\!R^3$$

the system is reachable with reachability indices $r_1 = 2$, $r_2 = 1$. Since

$$C_0 = 0,\ C_1 = \text{image}\begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},\ C_2 = C_3 = \text{image}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I\!R^3$$

the system is controllable and $q = 2$.

Deadbeat gains can be calculated using Theorem 6.1. One can take

$$S_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

thus obtaining, recursively,

$$L_1 = \begin{bmatrix} \beta & 1+\beta & 0 \\ \alpha & \alpha & 1 \end{bmatrix}, \quad L_2 = L_3 = \begin{bmatrix} 1 & 2 & 0 \\ \alpha & \alpha & 1 \end{bmatrix}$$

for any real numbers α and β. Any and all deadbeat gains are given as

$$L = \begin{bmatrix} 1 & 2 & 0 \\ \alpha & \alpha & 1 \end{bmatrix}.$$

The closed-loop system (6.1), (6.2) is described by

$$A - BL = \begin{bmatrix} -1 & -1 & 0 \\ 1 & 1 & 0 \\ -\alpha & -\alpha & 0 \end{bmatrix},$$

which is a nilpotent matrix with index $q = 2$. Any initial state is driven to the controllability subspace $C1$ in one step and then to the origin in the second step.

Let us now transform (6.1) to the reachability standard form. An appropriate similarity transformation T is found to be

$$T = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

It allows calculating a deadbeat gain as the LQ regulator gain that is optimal with respect to $C = T$ and $D = 0$.

Since rank $B = 2$, the conditions of Theorem 6.2 are all satisfied. The algebraic Riccati equation (6.7) has a unique symmetric non-negative definite solution

$$X = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The resulting LQ regulator gain (6.6),

$$L = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

is indeed a deadbeat gain, corresponding to $\alpha = 0$. The other deadbeat gains, however, cannot be obtained using this approach.

6.5. Conclusion

Deadbeat control and LQ regulation in discrete-time systems, two control strategies that are so different in nature, are in fact related. It has been shown that a deadbeat control law can be obtained by solving a particular LQ regulator problem, at least for reachable systems. This demonstrates the flexibility offered by the LQ regulator design.

The LQ optimal regulator gain is unique, whereas the deadbeat feedback gains are not. Only one deadbeat gain is LQ optimal. An alternative construction of such a gain, based on solving an algebraic Riccati equation, is thus available.

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8. PŘÍLOHY

8.1. Příklady k učebním textům

1. Určete všechny zpětnovazební regulátory s ryzím racionálním přenosem $R(s)$, které asymptoticky stabilizují minimální realizaci přenosu

   $$S(s)=\frac{s+1}{s}.$$

   Je mezi nimi proporcionální regulátor $R(s)=1$?

2. Určete všechny zpětnovazební regulátory s ryzím racionálním přenosem $R(s)$, které asymptoticky stabilizují minimální realizaci přenosu

   $$S(s)=\frac{s}{s+1}.$$

   Je mezi nimi regulátor $R(s)=0$?

3. Určete všechny zpětnovazební regulátory s ryzím racionálním přenosem $R(s)$, které asymptoticky stabilizují minimální realizaci přenosu

   $$S(s)=\frac{s}{s-1}.$$

4. Určete stavovou reprezentaci $(\overline{A},\overline{B},\overline{C},0)$ řádu 1 všech striktně ryzích regulátorů, které asymptoticky stabilizují systém $\dot{x}=x+u,\ y=x+u$.
5. Dán diskrétní systém s přenosem

   $$S(z)=\frac{1}{(z-1)^2}.$$

   Určete přenos regulátoru, který zajistí nejkratší odezvy y a e na jednotkový impuls, přivedený na vstupy r a d.

6. Dán systém, který je minimální realizací přenosu

   $$S(z)=\frac{1}{s-2}.$$

   Určete přenos regulátoru, který zpětnovazební systém stabilizuje a minimalizuje normu H_2 jeho přenosu H_C mezi vstupem r a výstupem y.

7. Dán diskrétní systém, který je minimální realizací přenosu

$$S(z) = \frac{1}{z-2}.$$

Určete přenos regulátoru, který zpětnovazební systém stabilizuje a minimalizuje normu l_1 jeho přenosu H_S mezi vstupem r a odchylkou e.

8. Dán systém $x_{k+1} = Ax_k + Bu_k$ s maticemi

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Určete všechna zesílení L zpětnovazebního zákona řízení $u_k = -Lx_k$ x_k tak, aby výsledný systém převedl libovolný stav x_0 do nulového stavu $x_k = 0$ za nejmenší počet kroků K. Kolik je K?

9. Dán systém $x_{k+1} = Ax_k + Bu_k$ s maticemi

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

a kvadratické kritérium

$$J = \| Cx_k + Du_k \|_2$$

s maticemi

$$C = [1 \quad 0], \quad D = 0.$$

Určete zpětnovazební zákon řízení $u_k = -Lx_k$, který minimalizuje J pro každý počáteční stav x_0.

10. Dán systém $x_{k+1} = Ax_k + Bu_k$ s maticemi

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Určete zákon řízení $u_k = -Lx_k$, který převede libovolný počáteční stav do nuly za nejkratší počet kroků (a) přímým výpočtem a (b) minimalizací kvadratického kritéria.

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