INVESTICE DO ROZVOJE VZDĚLÁVÁNÍ

Automatic Control of Vehicular Platoons via 2-D Polynomial Approach

Učební texty k semináři

Autoři:

Prof. Michael Šebek (Czech Technical University in Prague)

Datum:

6. 4. 2012

Centrum pro rozvoj výzkumu pokročilých řídicích a senzorických technologií
CZ.1.07/2.3.00/09.0031

TENTO STUDIJNÍ MATERIÁL JE SPOLUFINANCOVÁN EVROPSKÝM SOCIÁLNÍM FONDEM A STÁTNÍM ROZPOČTEM ČESKÉ REPUBLIKY

Contents

This text is an extended version of the conference paper: Šebek, M. and Hurák, Z.: 2-D polynomial approach to control of leader following vehicular platoons. In *Proceedings of the 18th IFAC World Congress,* 2011. Bologna: IFAC, 2011, p. 6017-6022. ISBN 978-3-902661-93-7.

Chapter 1

Introduction

The goal of this paper is to build a mathematical formalism needed to model and analyze an infinite platoon of vehicles following their leader, and demonstrate the elegance of the approach by solving a few classical platooning problems. Even though a platoon with an infinite number of vehicles constitutes an unrealistic model of reality, it can be used to infer the asymptotic properties of long but finite platoons. This approximation was proposed in 1970s by [18] and [3] and then reexamined three decades later by [10].

The problem of controlling a long platoon (also called string) of vehicles was intensively studied as early as in 1960s. Probably the first treatment was from the viewpoint of optimal control theory as reported by [16], which consisted in straightforward application of conventional LQ-optimal control design techniques for a MIMO state-space system having as state variables the deviations of inter-vehicular distances and vehicle velocities from some required values. The optimality criterion included not only the terms corresponding to the two types of state variables but also a term corresponding the deviations of the effective forces from a nominal value. [20] extended this optimality criterion by adding a term that corresponds to deviation of the vehicle absolute position from its scheduled value and another term corresponding to jerk (rate of change of the force) which is related to a passenger comfort. [17] defines the error variable in the state space model slightly differently compared to [16], the state variables corresponding to position are just deviations from the scheduled absolute positions. Provisions for weighting the distance between two vehicles are made by having the weighting matrix bidiagonal. Then they explore the structure of the LQ problem to provide a closed-form solution, breaking the problem down to solution of decoupled scalar second order problems.

The same problem was also approached by [18] approximating the long but finite string of vehicles by an infinitely long string. Invoking twosided z-transform, with the z variable corresponding to a shift in the index of the vehicle, the design problem of a conventional LQ-optimal state feedback appears in a modified form wherein the matrices appearing in the constituent

Riccati equations are no longer constant but rather polynomial matrices in the z variable. The optimal state feedback controller that is to be run independently at every vehicle then turns out to require information from all the vehicles in the string, even though the mutual influence of vehicles is diminishing as their distance (measured in the number of vehicles between them) is increasing. The necessity to have the global information about the string of vehicles makes this solution far from amenable to practical implementation. [3] elaborated on the same problem within the same setting of a bilateral z-transform but with some enforced various spatial constraint on the distributed controller. For instance they show that with just measurements of the distance from the vehicle ahead and the absolute (self)velocity of the vehicle, it is impossible to guarantee stability of the platoon. Basic reference on various stability concepts for vehicular platoons are given later in Chapter 6.

We were inspired by the critical analysis of [9] who show that the problem of distributed control of a string of vehicles as cast in the original papers by is ill-conditioned, that is, as the number of vehicles in the string grows, the string becomes more and more difficult to stabilize (and impossible to stabilize in the infinite case). Quite surprising that this issue went unnoticed for so long with these often cited papers.

Our approach is based on so called spatially invariant systems. This type of systems was studied in the late 1960s and early 1970s within a broader class of systems whose coefficients are functions of parameters. The right mathematical concept appeared to be that of linear systems over rings, because the coefficients in the state-space matrices and the coefficients in the transfer functions are elements of a ring. This broader class of systems also includes systems with delays or systems over integers. Among the pioneers in the area of linear systems over (commutative) rings were Kalman and his doctoral student [21] and [14]. Readable surveys were given by [23] and [11]. Specialization of these general results to spatially distributed systems was given in another survey paper by [12]. A few papers followed in the early 1980s such as [13] and Khargonekar [15], but the interest of the community into this field faded away towards the end of 1980s and throughout 1990s. Surprisingly, the field was revived around the beginning of the new millennium, through the papers by [1], [6],[25], [10] and [24], to name just a few, demanded by new technologies such as MEMS, adaptive optics, networked systems, low cost UAVs or mobile robots.

Distinguished feature of this paper is that while majority of the mentioned papers including [9] rely on state-space formalism, here the preference is given to input-output description, that is, models are given in the form of a fraction of two bivariate polynomials. This approach was first sketched in [8]. Major justification for this preference is availability of some promising analytical and numerical tools from multidimensional system theory, some of which were developed by the authors of this paper, for instance [4], [29], [32]. Another feature of this paper is that the platoon is assumed to have a leader, that is, the cars are indexed by natural numbers. Joint unilateral Laplace and z-transform, formally denoted here as $\mathcal{LZ}_1$-transform, is used to model the problem at hand by a fraction of bivariate polynomials. This brings the platooning problem on the same ground as numerous problems in the broad and well studied domain of 2-D signals and systems.

Chapter 2

Platoon description

Semi-infinite one-dimensional platoon studied in the paper is shown in Fig. 2.1. The leading vehicle is labeled by 0 and the follow-up cars are numbered by $1, 2, \ldots$. The vehicles keep their original indices even when exchanging their positions. The leader is driven externally while the followers are controlled by the algorithms discussed in the paper.

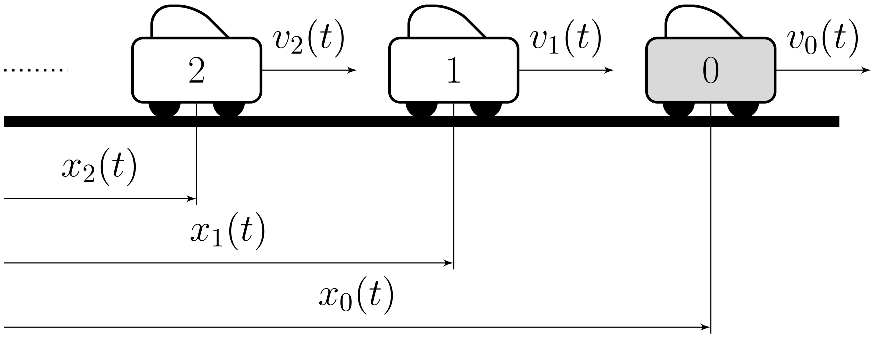

Figure 2.1: Platoon of vehicles with a leader.

Variables in the platoon, such as positions and velocities are described by spatial sequences of time functions

$$\{f(t, k)\} = f(t, 0), f(t, 1), f(t, 2), \ldots, \; t \in [0, \infty),$$

corresponding to the equally indexed vehicles.

Chapter 3

$\mathcal{LZ}_1$-transform and its properties

To prepare the ground, a joint unilateral Laplace and (shifted) unilateral z-transform denoted $\mathcal{LZ}_1$ is defined as

$$\mathcal{LZ}_1\left\{f(t,k)\right\} = \int_{0^-}^{\infty}\left(\sum_{k=1}^{\infty} f(t,k)z^{-k}\right)e^{-st}\,dt. \tag{3.1}$$

In contrast to the common z-transform definition, the discrete-space part of the $\mathcal{LZ}_1$-transform "starts" with the vehicle indexed by $k=1$. This keeps the leader outside the support allowing the $\mathcal{LZ}_1$ - transform to describe just the controlled vehicles. The movement of the leading vehicle then becomes a boundary condition.

The $\mathcal{LZ}_1$-transform of the sequence $\{f(t,k)\}$ expands[1] into

$$f(s,z) = \underbrace{f(s,1)}_{f_1(s)} z^{-1} + \underbrace{f(s,2)}_{f_2(s)} z^{-2} + \ldots \tag{3.2}$$

which is a formal power series in z^{-1} having polynomials or fractions in s as its coefficients.

A couple of $\mathcal{LZ}_1$-transform properties are listed here that are used in the paper. Their proofs are sketched in the Appendix.

Theorem 1 (*$\mathcal{LZ}_1$-transform of time derivatives*)**.**
Given spatial sequence of time functions $f(t,k)$ and its $\mathcal{LZ}_1$-transform $f(s,z)$, then

$$\mathcal{LZ}_1\left\{\frac{\partial f}{\partial t}\right\} = sf\left(s,z\right) - f_{0^-}\left(z\right), \tag{3.3}$$

$$\mathcal{LZ}_1\left\{\frac{\partial^2 f}{\partial t^2}\right\} = s^2 f\left(s,z\right) - sf_{0^-}\left(z\right) - \dot{f}_{0^-}\left(z\right), \tag{3.4}$$

[1] We are rather careless with the notation here as both the original spatiotemporal signal and its transform are labeled with the same letter, being distinguished only by their arguments (t,k) vs. (s,z).

assuming that the derivatives exist. Here

$$f_{0^-}(z) = \sum_{k=1}^{\infty} f(0^-, k) z^{-k}, \tag{3.5}$$

$$\dot{f}_{0^-}(z) = \sum_{k=1}^{\infty} \dot{f}(0^-, k) z^{-k}, \tag{3.6}$$

are $\mathcal{Z}_1$-transforms of the spatial sequences of (pre)initial conditions $f(0^-, k)$ and $\dot{f}(0^-, k)$, respectively.

Theorem 2 (*$\mathcal{L}\mathcal{Z}_1$-transform of space shift*)**.**
Given spatial sequence of time functions $f(t, k)$ and its $\mathcal{L}\mathcal{Z}_1$-transform $f(s, z)$, then

$$\mathcal{L}\mathcal{Z}_1 \{f(t, k-1)\} = z^{-1} f(s, z) + z^{-1} f_0(s), \tag{3.7}$$

where

$$f_0(s) = \int_{0^-}^{\infty} f(t, 0) e^{-st} dt \tag{3.8}$$

is the $\mathcal{L}$-transform of the function related to the leader.

Chapter 4

Platoon as a general 2-D System

Platoons and their controls are modeled here in a compact general form using fractions of real bivariate polynomials. The two variables are denoted as s and z, corresponding to time and the spatial index of the vehicle, respectively. A variety of platoons is described by the general 2-D plant

$$a(s,z)y(s,z) = b(s,z)u(s,z) + c(s,z). \tag{4.1}$$

Here $y(s,z)$ and $u(s,z)$ stand for $\mathcal{LZ}_1$-transforms of the plant output and input, respectively. Writing them as formal power series in z^{-1} with rational coefficients in s

$$y(s,z) = y_1(s)z^{-1} + y_2(s)z^{-2} + \ldots, \tag{4.2}$$

$$u(s,z) = u_1(s)z^{-1} + u_2(s)z^{-2} + \ldots, \tag{4.3}$$

nicely reveals that particular coefficients $y_k(s)$ and $u_k(s)$ represent the local output and local input at the position number k.

Furthermore, $a(s,z)$ and $b(s,z)$ are 2-D polynomials encountered in the plant transfer function, while $c(s,z)$ is a 2-D polynomial or fraction incorporating the information about the initial and boundary conditions in the plant. Their roles become evident from rewriting (4.1) into

$$y(s,z) = \frac{b(s,z)}{a(s,z)}u(s,z) + \frac{c(s,z)}{a(s,z)}. \tag{4.4}$$

Correspondingly, a general 2-D controller

$$p(s,z)u(s,z) = q(s,z)e(s,z) + d(s,z), \tag{4.5}$$

which is driven by error signal

$$e(s,z) = y_{\text{ref}}(s,z) - y(s,z), \tag{4.6}$$

covers a miscellany of control schemes. The role of the polynomials $p(s,z)$, $q(s,z)$ and $d(s,z)$ is clear[1] from

$$u(s,z) = \frac{q(s,z)}{p(s,z)}e(s,z) + \frac{d(s,z)}{p(s,z)}. \tag{4.7}$$

[1]Throughout the paper, the influence of initial and boundary conditions in the controller is usually neglected, which sets $d(s,z) = 0$.

Use of these general 2-D models is now demonstrated on typical control policies.

Example 1: Predecessor Following Control. Consider a platoon of identical vehicles, each governed by a simple double integrator equation, where for every vehicle the distance to its predecessor is measured and used for control. In time and space, such a platoon is modeled by the equations (for $t \in [0, \infty]$, $k = 1, 2, 3, \ldots$)

$$\begin{aligned} \ddot{x}(t,k) &= \frac{1}{m} u(t,k), \\ r(t,k) &= x(t,k-1) - x(t,k), \end{aligned} \tag{4.8}$$

where the quantities $x(t,k)$, $u(t,k)$ and $r(t,k)$ stand for the position of the k-th vehicle, its control input (driving force) and its distance from the $(k-1)$-th vehicle, its predecessor, respectively. Naturally, the whole sequences $\{x(t,k)\}$, $\{u(t,k)\}$ and $\{r(t,k)\}$ describe the positions of all the vehicles, all the driving forces (local inputs) and all the distances between the neighboring vehicles, respectively.

To complete the model, some initial as well as boundary conditions must be known. These are the initial positions $x(0^-, k) = x_{0^-}(k)$ and the velocities $\dot{x}(0^-, k) = \dot{x}_{0^-}(k)$ for all $k = 1, 2 \ldots$ as well as the leader's position $x(t, 0) = x_0(t)$ for all $t \in [0, \infty)$.

The $\mathcal{LZ}_1$-transform turns (4.8) into

$$\begin{aligned} x(s,z) &= \frac{1}{ms^2} u(s,z) + \frac{1}{s} x_{0^-}(z) + \frac{1}{s^2} \dot{x}_{0^-}(z), \\ r(s,z) &= (z^{-1} - 1) x(s.z) + z^{-1} x_0(s). \end{aligned} \tag{4.9}$$

Putting this together yields

$$\begin{aligned} ms^2 r(s,z) = (z^{-1} - 1) u(s,z) &+ ms(z^{-1} - 1) x_{0^-}(z) \\ &+ m(z^{-1} - 1) \dot{x}_{0^-}(z) + ms^2 z^{-1} x_0(s), \end{aligned}$$

which matches the general format of (4.1) with the output $y(s,z) = r(s,z)$ and the corresponding polynomials

$$\begin{aligned} a(s,z) &= ms^2, \\ b(s,z) &= (z^{-1} - 1), \\ c(s,z) &= ms(z^{-1} - 1) x_{0^-}(z) + \\ &\quad + m(z^{-1} - 1) \dot{x}_{0^-}(z) + ms^2 z^{-1} x_0(s). \end{aligned} \tag{4.10}$$

A natural strategy is to control each vehicle locally by a controller operating on the error of the distance to the predecessor from its desired reference value.

When all the local controllers are identical, the $\mathcal{L}$-transform yields

$$p(s)u_k(s) = q(s)\left(r_{\mathrm{ref},k}(s) - r_k(s)\right), \tag{4.11}$$

where the role of initial conditions is neglected. Global controller, which can be viewed as a sequence of local controllers, fits into general 2-D format (7.5) with

$$\begin{aligned} e(s,z) &= r_{\mathrm{ref},k}(s) - r_k(s), \\ p(s,z) &= p(s), q(s,z) = q(s), d(s,z) = 0. \end{aligned} \tag{4.12}$$

Example 2: Leader Following Control. As another example, consider again the platoon above, where now for every vehicle its distance to the leader is measured and used for control. Such a platoon is described by the equations (for $t \in [0,\infty],\ k = 1,2,3,\ldots$)

$$\begin{aligned} \ddot{x}(t,k) &= \frac{1}{m}u(t,k), \\ w(t,k) &= x(t,0) - x(t,k), \end{aligned} \tag{4.13}$$

where $w(t,k)$ stands for the distance between the k-th and the leading ($0th$) vehicle. The initial and the boundary conditions are as above. Using $\mathcal{LZ}_1$-transform, (4.13) becomes

$$x(s,z) = \frac{1}{ms^2}u(s,z) + \frac{1}{s}x_{0^-}(z) + \frac{1}{s^2}\dot{x}_{0^-}(z), \tag{4.14}$$

$$w(s,z) = \frac{z^{-1}}{1-z^{-1}}x_0(s) - x(s,z), \tag{4.15}$$

from which finally

$$\begin{aligned} &-(z^{-1}-1)ms^2w(s,z) = (z^{-1}-1)u(s,z) + \\ &\quad ms(z^{-1}-1)x_{0^-}(z) + m(z^{-1}-1)\dot{x}_{0^-}(z)ms^2z^{-1}x_0(s). \end{aligned}$$

Matching this to (4.1) with the output $y(s,z) = w(s,z)$ gives

$$\begin{aligned} a(s,z) &= (1-z^{-1})ms^2, \\ b(s,z) &= (z^{-1}-1), \\ c(s,z) &= ms(z^{-1}-1)x_{0^-}(z) \\ &\qquad + m(z^{-1}-1)\dot{x}_{0^-}(z) + ms^2z^{-1}x_0(s). \end{aligned} \tag{4.16}$$

When every vehicle controller is fed by the deviation of its distance to the leader from the desired distance and their dynamics are identical, they are driven by

$$p(s)u_k(s) = q(s)(w_{\mathrm{ref},k}(s) - w_k(s)), \tag{4.17}$$

where the role of initial conditions is again ignored. Global controller then fits into the general 2-D format (7.5) with

$$\begin{aligned} e(s,z) &= w_{\mathrm{ref},k}(s) - w_k(s), \\ p(s,z) &= p(s), q(s,z) = q(s), d(s,z) = 0. \end{aligned} \tag{4.18}$$

Example 3: Constant Time-Headway Policy. Yet another example requires not only the measurements of the inter-vehicular distances but also the absolute velocities because the desired spacing is now depending on the velocity via

$$r_{\mathrm{ref}}(t,k) = \bar{r}_0 + \bar{r}\dot{x}(t,k). \tag{4.19}$$

This requirement is called the constant time-headway policy. After $\mathcal{LZ}_1$-transform, (4.19) turns into

$$r_{\mathrm{ref}}(s,z) = \frac{z^{-1}}{1-z^{-1}}\frac{\bar{r}_0}{s} + \bar{r}sx(s,z) - \bar{r}x_{0^-}(z). \tag{4.20}$$

The platoon is then described (for $t \in [0,\infty],\ k = 1,2,3,\ldots$) by

$$\begin{aligned} \ddot{x}(t,k) &= \frac{1}{m}u(t,k), \\ \bar{y}(t,k) &= x(t,k-1) - x(t,k) - \bar{r}\dot{x}(t,k), \end{aligned} \tag{4.21}$$

with the "output" $\bar{y}(t,k)$ standing for the inter-vehicular distance reduced by the velocity-dependent factor. The initial and boundary conditions are as above and the $\mathcal{LZ}_1$-transform produces

$$\begin{aligned} x(s,z) &= \frac{1}{ms^2}u(s,z) + \frac{1}{s}x_{0^-}(z) + \frac{1}{s^2}\dot{x}_{0^-}(z), \\ \bar{y}(s,z) &= (z^{-1} - 1 - \bar{r}s)x(s,z) + z^{-1}x_0(s) + \bar{r}x_{0^-}(z), \end{aligned}$$

and simple algebra gives rise to the single equation

$$\begin{aligned} ms^2y(s,z) = &\left(z^{-1} - \bar{r}s - 1\right)u(s,z) \\ &+ ms(z^{-1}-1)x_{0^-}(z) \\ &+ m(z^{-1}-1)\dot{x}_{0^-}(z) + ms^2z^{-1}x_0(s). \end{aligned}$$

Once again, the equation matches the general format (4.1), now with $y(s,z) = \bar{y}(s,z)$ and

$$\begin{aligned} a(s,z) &= ms^2, \\ b(s,z) &= \left(z^{-1} - \bar{r}s - 1\right), \\ c(s,z) &= ms(z^{-1}-1)x_{0^-}(z) \\ &\quad + m(z^{-1}-1)\dot{x}_{0^-}(z) + ms^2z^{-1}x_0(s). \end{aligned} \tag{4.21}$$

A convenient controller is here

$$p(s)u(s,z) = q(s)\left(\bar{y}_{\mathrm{ref}}(s,z) - \bar{y}(s,z)\right), \tag{4.22}$$

where

$$\bar{y}_{\mathrm{ref}}(s,z) = \frac{z^{-1}}{1-z^{-1}}\frac{\bar{r}_0}{s}. \tag{4.23}$$

It is actually driven by the regulation error as

$$\begin{aligned} e(s,z) &= \bar{y}_{\mathrm{ref}}(s,z) - \bar{y}(s,z) \\ &= \frac{z^{-1}}{1-z^{-1}}\frac{\bar{r}_0}{s} - \left(z^{-1} - 1 - \bar{r}s\right)x(s,z) \\ &\quad - z^{-1}x_0(s) - \bar{r}x_{0^-}(z) \\ &= \frac{z^{-1}}{1-z^{-1}}\frac{\bar{r}_0}{s} + \bar{r}sx(s,z) - \bar{r}x_{0^-}(z) \\ &\quad - (z^{-1}-1)x(s,z) - z^{-1}x_0(s) \\ &= r_{\mathrm{ref}}(s,z) - r(s,z). \end{aligned} \tag{4.24}$$

Such a controller is, of course, again just a particular case of the general 2-D controller (7.5) with

$$p(s,z) = p(s), q(s,z) = q(s), d(s,z) = 0. \tag{4.25}$$

Chapter 5

Control for a general 2-D system

This paper investigates how the distributed control schemes aimed at following the leader and/or other reference commands scale with the growing number of vehicles. In the input-output setting, this goal is usually rephrased as a stability requirement for certain transfer functions.

Putting together the general 2-D plant and a 2-D controller equations (4.1) and (4.5), they implicitly relate certain variables that are "given" or "supplied from outside" to other variables that are to be controlled or at least taken into account. The "given" variables include the reference command $y_{\mathrm{ref}}(t,k)$ as well as the initial and boundary conditions in the plant. The conditions are $x(0^-,k) = x_{0^-}(k)$, $\dot{x}(0^-,k) = \dot{x}_{0^-}(k)$ and $x(t,0) = x_0(t)$ and are included in $c(s,z)$ through

$$c(s,z) = c_1\left(s,z\right)x_{0^-}\left(z\right) + c_2\left(s,z\right)\dot{x}_{0^-}\left(z\right) + c_3\left(s,z\right)x_0\left(s\right).$$

The initial and boundary conditions of the controller, expressed similarly by $d(s,z)$, are also part of the game.

The controlled or otherwise notable variables[1] naturally comprise the error $e(s,z)$ as the measure of quality, the plant output $y(s,z)$, as well as the plant input $u(s,z)$. Their explicit expressions, assuming $d(s,z)=0$, are

$$e(s,z) = \frac{a(s,z)p(s,z)}{\bar{m}(s,z)}y_{\mathrm{ref}}(s,z) - \frac{p(s,z)}{\bar{m}(s,z)}c(s,z), \tag{5.1}$$

$$y(s,z) = \frac{b(s,z)q(s,z)}{\bar{m}(s,z)}y_{\mathrm{ref}}(s,z) + \frac{p(s,z)}{\bar{m}(s,z)}c(s,z), \tag{5.2}$$

$$u(s,z) = \frac{a(s,z)q(s,z)}{\bar{m}(s,z)}y_{\mathrm{ref}}(s,z) + \frac{q(s,z)}{\bar{m}(s,z)}c(s,z), \tag{5.3}$$

where we have denoted the common denominator by

$$a(s,z)p(s,z) + b(s,z)q(s,z) = \bar{m}(s,z). \tag{5.4}$$

[1] Notable variables not appearing in (4.1) and (4.5) can be computed from the particular platoon equations. So in Example 1, one gets the positions $x(s,z)$ from $r(s,z)$ via (4.8), etc.

The relations (5.1-5.3) consist of all the closed-loop transfer functions from the given variables to the controlled or notable variables.
Common denominator of all the transfer functions – the polynomial $\bar{m}(s,z)$ – arises from (5.4). Given the plant, i.e. $a(s,z)$ and $b(s,z)$, various right hand sides can be achieved by choosing the controller, i.e. $p(s,z)$ and $q(s,z)$. Reversely, given the plant and the polynomial $\bar{m}(s,z)$, (5.4) can be solved as a 2-D polynomial equation.
The right-hand side must vanish at all common zeros of the left-hand side polynomials $a(s,z)$ and $b(s,z)$. If the common zeros are stable, a stable polynomial $\bar{m}(s,z)$ can be achieved. If they are unstable, so is every $\bar{m}(s,z)$. See [27, 28, 30] or [31] for more on 2-D polynomial equations.

Chapter 6

2-D BIBO stability and string stability

Two concepts of stability appear in the platooning literature: 2-D BIBO stability and string stability.
According to [12](Theorem 4.3, pp. 126), a spatially distributed 2-D system with a coprime transfer $f(s,z) = b(s,z)/a(s,z)$ is BIBO stable if

$$\begin{aligned} & a(s, e^{j\omega}) \neq 0 \\ & \forall s \in \mathbb{C}, \omega \in \mathbb{R} : \Re(s) \leq 0, \omega \in [0, 2\pi] \end{aligned} \tag{6.1}$$

In other words, if it is a stable polynomial in s after substituting for z any complex number from the unit circle.
Note that the stability condition above is not necessary. The stability can sometimes be saved by nonessential singularities of the second kind at the distinguished stability boundary (see [7]). To avoid this subtlety, all relevant coprime transfer functions are required to have a stable denominator (not vanishing on the distinguished boundary of the stability domain). Every polynomial satisfying (6.1) is called 2-D stable.
All the transfer functions in (5.1)-(5.3) have the same denominator $\bar{m}(s,z)$. If it is stable, then all the transfer functions are BIBO stable. Even if $\bar{m}(s,z)$ is not stable, its unstable factor may happen to cancel in relevant transfer functions. In some experiments, also the denominators of $y_{\mathrm{ref}}(s,z)$ or $c(s,z)$ can be unstable.
Most papers that appeared in this domain were from the early days based on the concept of a *string stability*, which was introduced by [5] under the name *asymptotic stability* and essentially means that spacing errors between neighboring vehicles (induced by disturbances, noises or changes in the reference signals) are not amplified when propagated down the platoon. When the ℓ_2 norm is used to measure the error signals, the necessary condition is $\left\| \frac{\hat{e}_i(s)}{\hat{e}_{i-}(s)} \right\|_\infty \leq 1$. This concept was later extended for nonlinear systems by [26], who actually coined the term *string stability*, and derived sufficiency conditions as well. One of the early interesting results referring to this notion of stability is by [19] who shows that it is impossible to achieve string stability when only measurements of relative distance from the vehicle ahead

are measured and PID controller is used locally. [22] later argues that not only PID but every linear controller is incapable of string-stabilizing a platoon with such a measurement configuration simply because the achievable $\mathcal{H}_\infty$ norm is always above 1. (Interestingly enough, when the relative distance from the vehicle ahead is measured as well as the absolute velocity of the vehicle, string stability can be achieved with proportional distance and velocity controllers). One last work that needs to be cited is a submitted paper by [2] which relates string stability describing microscopic behavior of the platoon and coherence (or rigidity) that is best viewed as a macroscopic property.

Chapter 7

Simulation Experiment

To compare different control strategies, a simulation experiment is conducted. At the beginning, the platoon is traveling at a constant speed $\dot{x}_{0^-}$ with the vehicles evenly spaced by r_{0^-}. These initial conditions are described by

$$\begin{aligned} x_{0^-}(z) &= -r_{0^-}\frac{z^{-1}}{\left(1-z^{-1}\right)^2} = -100\frac{z^{-1}}{\left(1-z^{-1}\right)^2}, \\ \dot{x}_{0^-}(z) &= \dot{x}_{0^-}\frac{z^{-1}}{1-z^{-1}} = 30\frac{z^{-1}}{1-z^{-1}}. \end{aligned} \tag{7.1}$$

The vehicles should maintain their original intervals $r_{\text{ref}} = r_{0^-}$, which is expressed by the reference command

$$r_{\text{ref}}(s,z) = r_{\text{ref}}(s)\frac{z^{-1}}{1-z^{-1}} = \frac{100}{s}\frac{z^{-1}}{1-z^{-1}}. \tag{7.2}$$

Besides, the vehicles should follow their leader. At the beginning, the leading vehicle is moving at the same constant speed, but then it slows down for a while and finally returns to its original velocity. This maneuver, serving as boundary condition, is described by

$$x_0(s) = \frac{30}{s^2} - \frac{10}{s^2}e^{-10s} + \frac{10}{s^2}e^{-15s}. \tag{7.3}$$

and visualized in Fig. 7.1.

7.1 Predecessor Following Control

In the predecessor following control, the plant and the controller are given as in Example 1 and (5.1)-(5.3) read

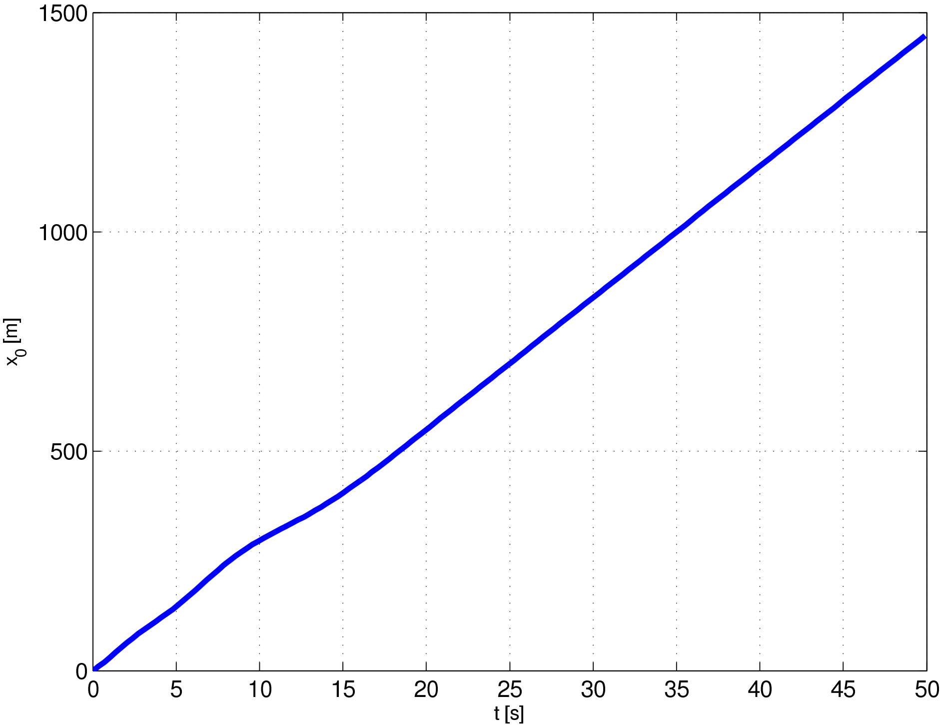

Figure 7.1: Leader's maneuver $x_0(t)$ to be followed.

$$
\begin{aligned}
e(s,z) = {} & \frac{ms^2 p(s)}{\bar{m}(s,z)} r_{\mathrm{ref}}(s,z) - \frac{p(s)\, ms^2 z^{-1}}{\bar{m}(s,z)} x_0(s) \\
& - \frac{p(s)\, ms\left(z^{-1}-1\right)}{\bar{m}(s,z)} x_{0^-}(z) \\
& - \frac{p(s)\, m\left(z^{-1}-1\right)}{\bar{m}(s,z)} \dot{x}_{0^-}(z), \\
r(s,z) = {} & \frac{\left(z^{-1}-1\right) q(s)}{\bar{m}(s,z)} r_{\mathrm{ref}}(s,z) + \frac{p(s)\, ms^2 z^{-1}}{\bar{m}(s,z)} x_0(s) \\
& + \frac{p(s)\, ms\left(z^{-1}-1\right)}{\bar{m}(s,z)} x_{0^-}(z) \\
& + \frac{p(s)\, m\left(z^{-1}-1\right)}{\bar{m}(s,z)} \dot{x}_{0^-}(z), \\
u(s,z) = {} & \frac{ms^2 q(s)}{\bar{m}(s,z)} r_{\mathrm{ref}}(s,z) - \frac{q(s)\, ms^2 z^{-1}}{\bar{m}(s,z)} x_0(s) \\
& - \frac{q(s)\, ms\left(z^{-1}-1\right)}{\bar{m}(s,z)} x_{0^-}(z) \\
& - \frac{q(s)\, m\left(z^{-1}-1\right)}{\bar{m}(s,z)} \dot{x}_{0^-}(z).
\end{aligned}
$$

The common closed-loop denominator

$$\bar{m}(s,z) = ms^2 p(s,z) + \left(z^{-1} - 1\right) q(s,z) \tag{7.4}$$

is clearly unstable as seen by substituting $z = 1$. In fact, this is caused by the unstable common zero $(s,z) = (0,1)$ in the plant transfer function. Even worse, no "unstable part" can be factored out of $\bar{m}(s,z)$ so that one cannot hope to cancel it. This results went unnoticed by [18] so that three decades later [10] show that the original scheme actually did not provide a stabilizing solution. It should be emphasized at this point that the concept of string stability used in most papers on strings or platoons of vehicles is different from the concept of 2-D BIBO stability considered here, since the latter admits spatial steps in reference signals whereas the former only assumes spatial impulses. In other words, the BIBO stability assumes bounded but persistent spatiotemporal signals whereas the string stability assumes local disturbance and studies how it propagates downstream.

Continued Example 1: Predecessor Following. When substituting particular experiment conditions, the above relations can be expanded into formal power series in z^{-1}. Particular terms of the series then describe behaviors of the corresponding vehicles. The terms are polynomial fractions in s which for increasing powers of z^{-1} can be shown to have increasing powers of the polynomial $\bar{m}(s,0) = ms^2 p(s) - q(s)$ in their denominators. Running the experiment with $m = 1$ and with a PD controller

$$\frac{q(s)}{p(s)} = -0.2 - s \tag{7.5}$$

results in the distances $r(t,k)$ and positions $x(t,k)$ shown in Fig. 7.2 and Fig. 7.3, respectively. The polynomial $\bar{m}(s,0) = s^2 + s + 0.2$ is stable so that each vehicle behaves locally well. Yet the spatial propagation of the behavior is rather ugly. This is a demonstration of string instability. Note that by increasing the damping of the system (by increasing the coefficient at the first power of s in the denominator polynomial), this nasty propagation can be attenuated. Yet the polynomial $\bar{m}(s,z)$ remains 2-D unstable, which demonstrates that the system is not BIBO stable and the platoon response will not scale well for a large number of vehicles.

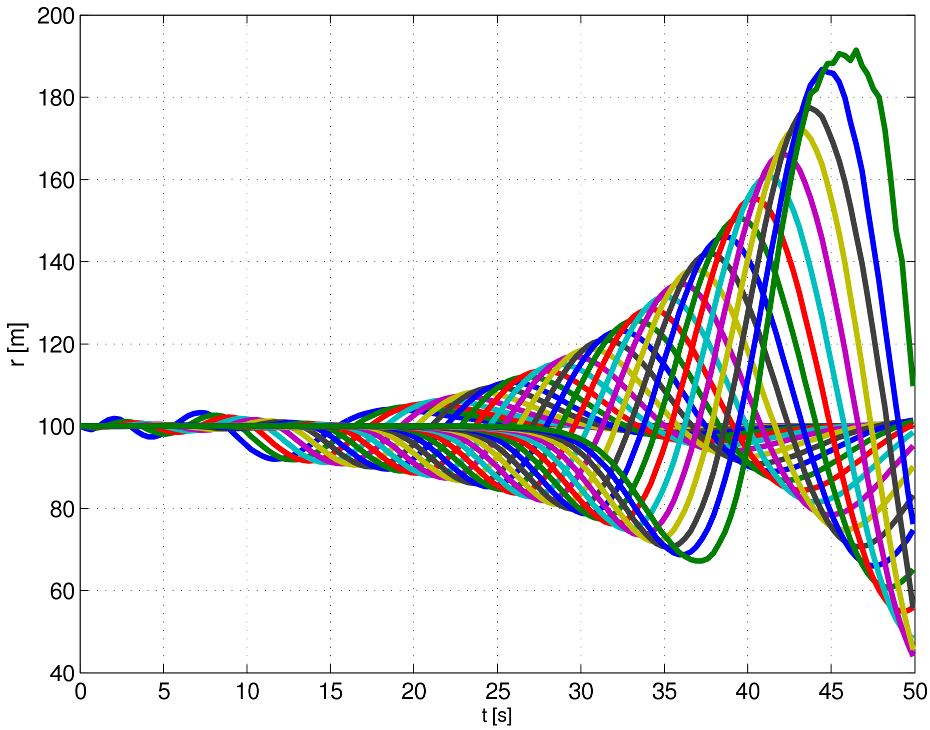

Figure 7.2: Distances $r(t, k)$ in predecessor following.

7.2 Leader Following Control

Leader following control described in Example 2 gives rise, via (5.4), to closed-loop common denominator polynomial

$$\bar{m}\left(s,z\right)=\left(1-z^{-1}\right)ms^{2}p\left(s\right)+\left(z^{-1}-1\right)q\left(s\right). \tag{7.6}$$

Even though $\bar{m}\left(s,z\right)$ is again 2-D unstable, it is factorable into the product of an unstable factor $\left(1-z^{-1}\right)$ with another factor

$$\bar{\bar{m}}\left(s\right)=ms^{2}p\left(s\right)-q\left(s\right). \tag{7.7}$$

The unstable factor $\left(1-z^{-1}\right)$ cancels in all the terms of $e(s,z)$ and $r(s,z)$ but unfortunately not in $u(s,z)$.

Continued Example 2: Leader Following. With the same experiment and local controller design (7.5), the results are quite different. Inter-vehicular distances $r(t, k)$ are identical for all vehicles and hence overlapping in Fig. 7.4. No amplifying propagation is observed in positions $x(t, k)$ shown in Fig. 7.5. And yet the system is not 2-D BIBO stable when the control variable is considered.

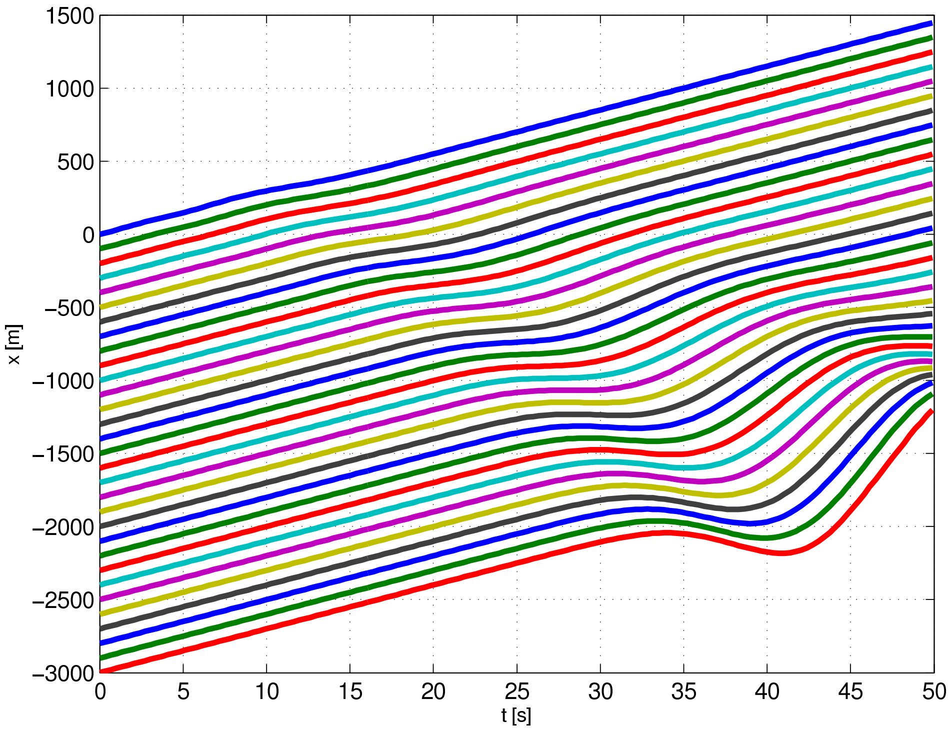

Figure 7.3: Positions $x(t,k)$ in predecessor following.

7.3 Constant Time-Headway Policy

With the constant time-headway policy (Example 3), the desired distances between the vehicles result from the policy parameters $\bar{r}_0, \bar{r}$ in (4.19). The common closed-loop denominator (5.4)

$$\bar{m}(s,z) = ms^2 p(s) + \left(z^{-1} - \bar{r}s - 1\right) q(s), \tag{7.8}$$

is again 2-D unstable as putting $z = 1$ makes it $\bar{m}(s,1) = s\,(msp(s) - \bar{r}q(s))$, a 1-D unstable polynomial.

Continued Example 3: Constant time-headway. For the policy parameters $\bar{r}_0 = 10, \bar{r} = 3$, resulting inter-vehicular distances $r(t,k)$ and positions $x(t,k)$ are shown in Fig. 7.6 and Fig. 7.7, respectively.

7.4 Spatial IIR controller

All the controllers mentioned so far were of local nature having $p(s,z) = p(s), q(s,z) = q(s)$ free of the space operator z^{-1}. Now consider a controller governed by

$$(u_k(s) - u_{k-1}(s))\, p(s) = q(s)\, e_k(s)\,.$$

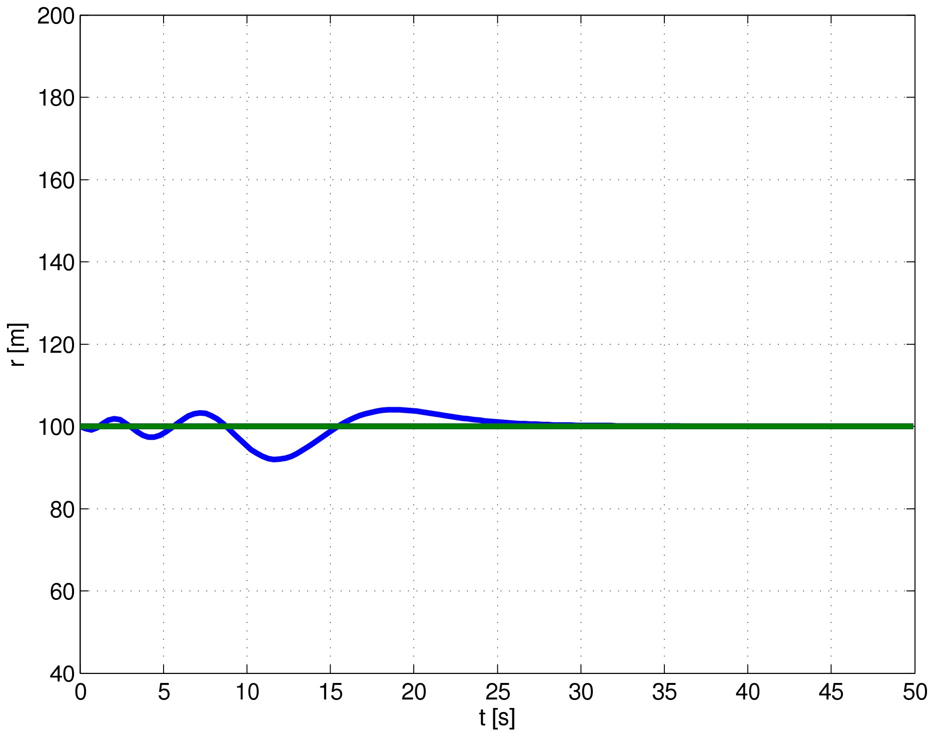

Figure 7.4: Leader following: distances $r(t, k)$.

Its transform

$$\left(1 - z^{-1}\right) p\left(s\right) u\left(s, z\right) = q\left(s\right) e\left(s, z\right) + z^{-1} p\left(s\right) u_0\left(s\right)$$

clearly matches (7.5) for

$$\begin{aligned} p\left(s, z\right) = \left(1 - z^{-1}\right) p\left(s\right), q\left(s, z\right) = q\left(s\right), \\ d\left(s, z\right) = z^{-1} u_0\left(s\right). \end{aligned} \tag{7.9}$$

This controller possesses a spatially (semi-)infinite impulse response (IIR), that is, all the local controller instances use the outcomes of the predecessors. The resulting closed-loop common denominator reads as (7.6)

$$\bar{m}\left(s, z\right) = \left(1 - z^{-1}\right) m s^2 p\left(s\right) + \left(z^{-1} - 1\right) q\left(s\right). \tag{7.10}$$

Hence it is 2-D unstable but factorable into the product of unstable and stable factors

$$\bar{m}\left(s, z\right) = \left(1 - z^{-1}\right) \left(m s^2 p\left(s\right) + q\left(s\right)\right) \tag{7.11}$$

Not only that this denominator is identical to (7.6) but also its unstable part cancels in some closed-loop transfer functions but not in the one relating the references (and disturbances) and the control variables (outputs of the controllers). The the overall performance then closely resemble leader following control.

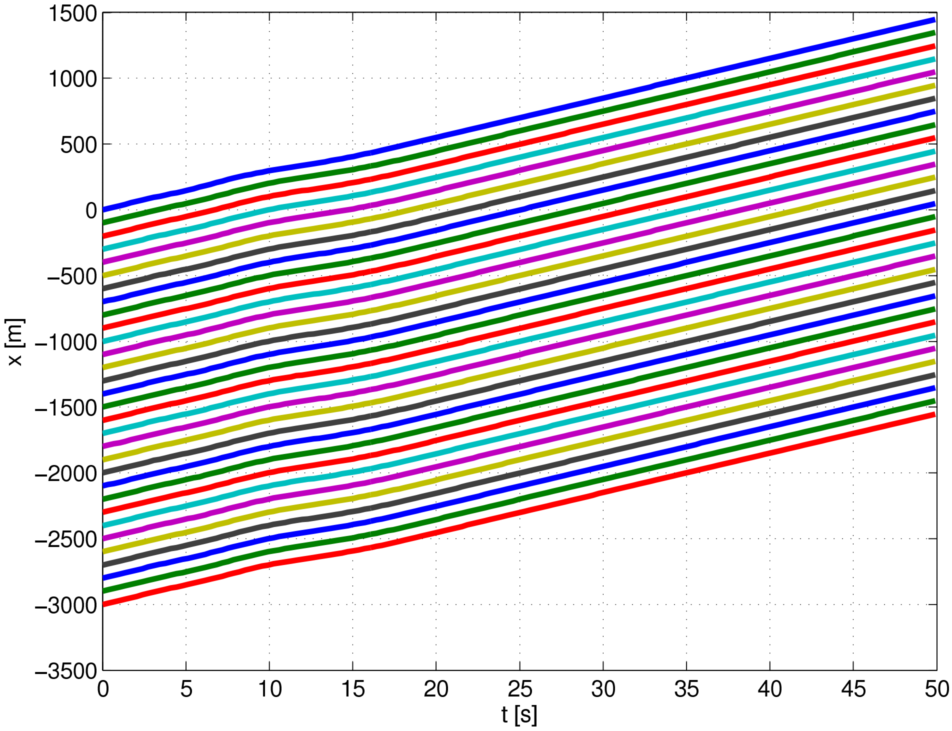

Figure 7.5: Leader following: positions $x(t, k)$.

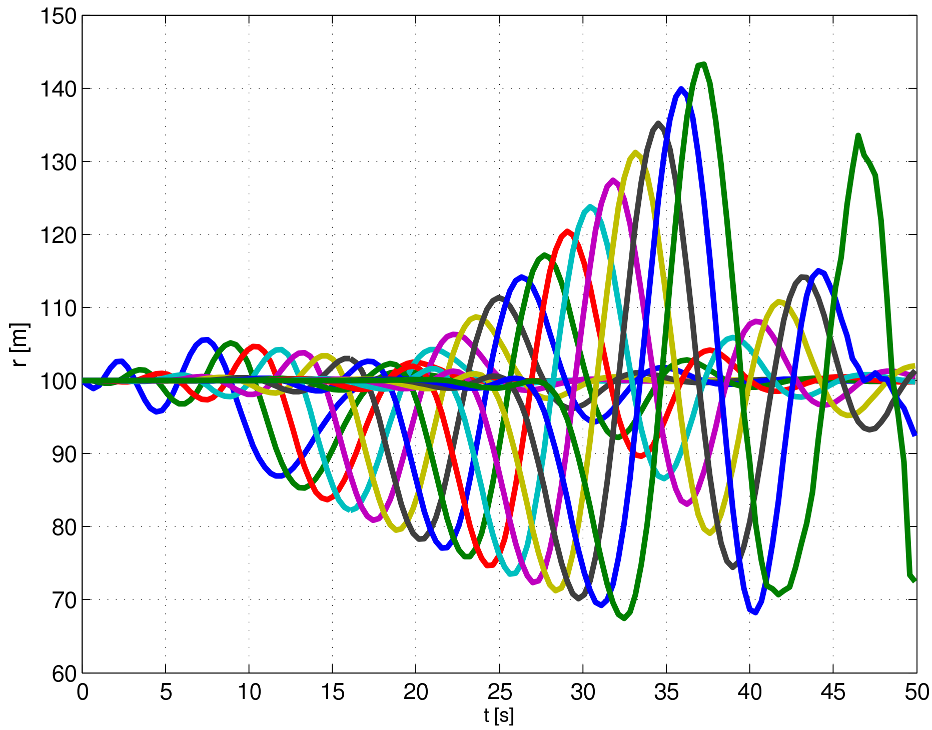

Figure 7.6: Constant time-headway policy: distances $r(t, k)$.

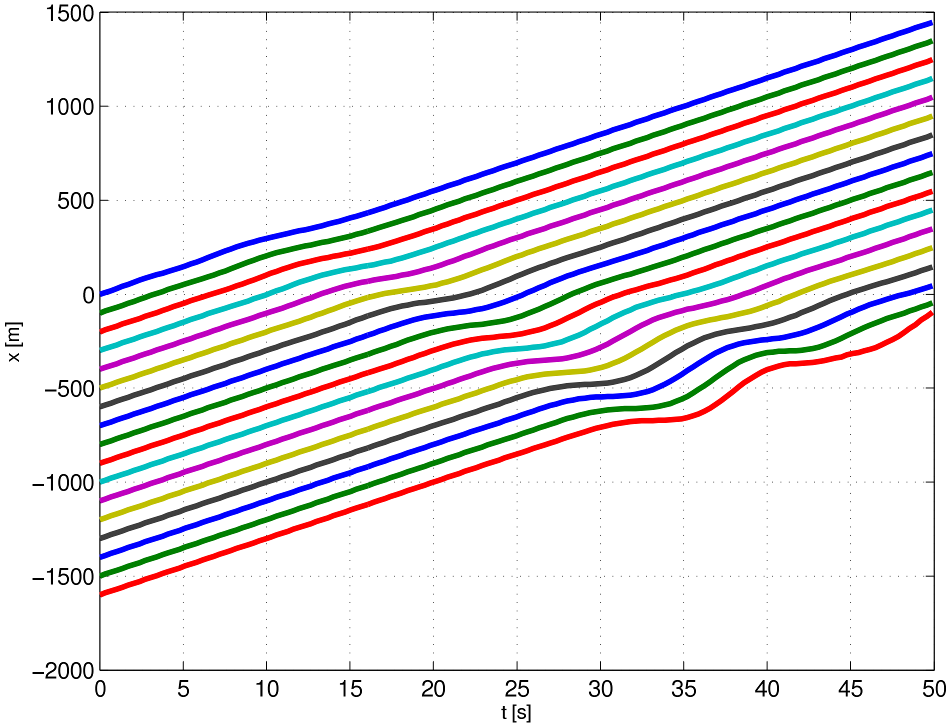

Figure 7.7: Constant time-headway policy: positions $x(t, k)$.

Chapter 8

Conclusions

The paper introduces a new formalism to the control of semi-infinite platoons of vehicles following their leader and it demonstrates its elegance by solving a few classical platooning problems. It is based on 2-D polynomials and their fractions resulting from a joint unilateral Laplace and z-transform – named here $\mathcal{LZ}_1$-transform. This makes it possible to model variety of platoons and controllers in a unified manner as well as to apply diverse control policies such as predecessor and leader following, constant time-headway etc. It was shown that both the string instability and 2-D BIBO instability can be detected in the proposed framework. The tools developed here are now ready for use in further research.

Appendix: Proofs of $\mathcal{LZ}_1$-Transform Properties

Proof of Theorem 1:
Swapping integration with summation in the $\mathcal{LZ}_1$-transform definition (3.1) yields

$$\mathcal{LZ}_1\left\{\frac{\partial f}{\partial t}\right\} = \int\limits_{0^-}^{\infty}\left(\sum_{k=1}^{\infty}\frac{\partial f\left(t,k\right)}{\partial t}z^{-k}\right)e^{-st}dt$$
$$= \sum_{k=1}^{\infty}\left(\int\limits_{0^-}^{\infty}\frac{\partial f\left(t,k\right)}{\partial t}e^{-st}dt\right)z^{-k}$$

Applying integration per partes with the fact that $\left(e^{-st}\right)' = -se^{-st}$ gives rise to

$$\int\limits_{0^-}^{\infty}\frac{\partial f\left(t,k\right)}{\partial t}e^{-st}dt = \left[f\left(t,k\right)e^{-st}\right]_{0^-}^{\infty}$$
$$-\int\limits_{0^-}^{\infty}f\left(t,k\right)\left(-se^{-st}\right)dt$$
$$= -f\left(0^-,k\right) + s\int\limits_{0^-}^{\infty}f\left(t,k\right)e^{-st}dt$$

Hence,

$$\begin{aligned}
&\mathcal{LZ}_1\left\{\frac{\partial f(t,k)}{\partial t}\right\} = \\
&= \sum_{k=1}^{\infty}\left(\int_{0^-}^{\infty}\frac{\partial f(t,k)}{\partial t}e^{-st}dt\right)z^{-k} \\
&= \sum_{k=1}^{\infty}\left(-f\left(0^-,k\right)+s\int_{0^-}^{\infty}f(t,k)\,e^{-st}dt\right)z^{-k} \\
&= s\int_{0^-}^{\infty}\left(\sum_{i=0}^{\infty}f(t,k)\,z^{-k}\right)e^{-st}dt-\sum_{k=1}^{\infty}f\left(0^-,k\right)z^{-k} \\
&= sf(s,z)-f_{0^-}(z)
\end{aligned}$$

The second derivative case can be proven alike by repeating the procedure of integration per partes. □

Proof of Theorem 2:
By the $\mathcal{LZ}_1$-transform definition

$$\mathcal{LZ}_1\{f(t,k-1)\} = \int_{0^-}^{\infty}\left(\sum_{k=1}^{\infty}f(t,k-1)\,z^{-k}\right)e^{-st}dt$$

When substituting $i=k-1$ and hence $k=i+1$ into the integrand, we have

$$\begin{aligned}
\sum_{k=1}^{\infty}f(t,k-1)\,z^{-k} &= \sum_{i=0}^{\infty}f(t,i)\,z^{-i-1} \\
&= z^{-1}\sum_{i=0}^{\infty}f(t,i)\,z^{-i} \\
&= z^{-1}\left(f(t,0)+\sum_{i=1}^{\infty}f(t,i)\,z^{-i}\right) \\
&= z^{-1}f(t,0)+z^{-1}\sum_{i=0}^{\infty}f(t,i)\,z^{-i}
\end{aligned}$$

Finally

$$
\begin{aligned}
&\mathcal{L}\mathcal{Z}_1\left\{f\left(t,k-1\right)\right\} = \\
&\quad = \int\limits_{0^-}^{\infty}\left(\sum_{k=1}^{\infty} f\left(t,k-1\right)z^{-k}\right)e^{-st}dt \\
&\quad = \int\limits_{0^-}^{\infty} z^{-1}\left(f\left(t,0\right) + \sum_{i=0}^{\infty} f\left(t,i\right)z^{-i}\right)e^{-st}dt \\
&\quad = z^{-1}\int\limits_{0^-}^{\infty} f\left(t,0\right)e^{-st}dt \\
&\qquad + z^{-1}\int\limits_{0^-}^{\infty}\left(\sum_{i=1}^{\infty} f\left(t,i\right)z^{-i}\right)e^{-st}dt \\
&\quad = z^{-1}f_0\left(s\right) + z^{-1}f\left(s,z\right)
\end{aligned}
$$

which proves the theorem. □

Bibliography

[1] B. Bamieh, F. Paganini, and M.A. Dahleh. Distributed control of spatially invariant systems. *Automatic Control, IEEE Transactions on*, 47(7):1091–1107, 2002.

[2] Bassam Bamieh, Mihailo R. Jovanovic, Partha Mitra, and Stacy Patterson. Coherence in Large-Scale networks: Dimension dependent limitations of local feedback. February 2009.

[3] Kai-Ching Chu. Decentralized control of High-Speed vehicular strings. *Transportation science*, 8(4):361–384, November 1974.

[4] B. Cichy, P.Augusta, E. Rogers, K. Gałkowski, and Z. Hurák. On the control of distributed parameter systems using a multidimensional systems setting. *Mechanical Systems and Signal Processing*, 22:1566–1581, October 2008.

[5] R.L. Cosgriff. The asymptotic approach to traffic dynamics. *Systems Science and Cybernetics, IEEE Transactions on*, 5(4):361–368, 1969.

[6] R. D'Andrea and G.E. Dullerud. Distributed control design for spatially interconnected systems. *Automatic Control, IEEE Transactions on*, 48(9):1478–1495, 2003.

[7] D. Goodman. Some stability properties of two-dimensional linear shift-invariant digital filters. *Circuits and Systems, IEEE Transactions on*, 24(4):201–208, 1977.

[8] Z. Hurák and M. Šebek. 2D polynomial approach to stability of platoons of vehicles. In *Proceedings of the 2nd IFAC Workshop on Distributed Estimation and Control in Networked Systems*, volume 2, Annecy, France, 2010.

[9] M.R. Jovanović and B. Bamieh. Lyapunov-based distributed control of systems on lattices. *Automatic Control, IEEE Transactions on*, 50(4):422–433, 2005.

[10] M.R. Jovanović and B. Bamieh. On the ill-posedness of certain vehicular platoon control problems. *Automatic Control, IEEE Transactions on*, 50(9):1307–1321, 2005.

[11] E. Kamen. Lectures on algebraic systems theory: Linear systems over rings. Contractor report 316, NASA, 1978.

[12] E. Kamen. Stabilization of Linear Spatially-Distributed Continuous-Time and Discrete-Time systems. In *Multidimensional Systems Theory: Progress, Directions and Open Problems in Multidimensional Systems*, Mathematics and Its Applications. D. Reidel Publishing Company, 1985.

[13] E. Kamen and P. Khargonekar. On the control of linear systems whose coefficients are functions of parameters. *Automatic Control, IEEE Transactions on*, 29(1):25–33, 1984.

[14] E. W. Kamen. On an algebraic theory of systems defined by convolution operators. *Theory of Computing Systems*, 9(1):57–74, March 1975.

[15] P. Khargonekar and E. Sontag. On the relation between stable matrix fraction factorizations and regulable realizations of linear systems over rings. *Automatic Control, IEEE Transactions on*, 27(3):627–638, 1982.

[16] W. Levine and M. Athans. On the optimal error regulation of a string of moving vehicles. *Automatic Control, IEEE Transactions on*, 11(3):355–361, 1966.

[17] S.M. Melzer and B.C. Kuo. A closed-form solution for the optimal error regulation of a string of moving vehicles. *Automatic Control, IEEE Transactions on*, 16(1):50–52, 1971.

[18] S.M. Melzer and B.C. Kuo. Optimal regulation of systems described by a countably infinite number of objects. *Automatica*, 7(3):359–366, May 1971.

[19] L. Peppard. String stability of relative-motion PID vehicle control systems. *Automatic Control, IEEE Transactions on*, 19(5):579–581, 1974.

[20] L. Peppard and V. Gourishankar. Optimal control of a string of moving vehicles. *Automatic Control, IEEE Transactions on*, 15(3):386–387, 1970.

[21] Y. Rouchaleau. *Linear, discrete time, finite dimensional, dynamical systems over some classes of commutative rings.* PhD thesis, Dept. of Operational Research, Stanford University, 1972.

[22] P. Seiler, A. Pant, and K. Hedrick. Disturbance propagation in vehicle strings. *Automatic Control, IEEE Transactions on*, 49(10):1835–1842, 2004.

[23] E. Sontag. Linear systems over commutative rings: A survey. *Ricerche di Automatica*, 7:1–34, 1976.

[24] G. Stein and D. Gorinevsky. Design of surface shape control for large two-dimensional arrays. *Control Systems Technology, IEEE Transactions on*, 13(3):422–433, May 2005.

[25] G.E. Stewart, D.M. Gorinevsky, and G.A. Dumont. Feedback controller design for a spatially distributed system: the paper machine problem. *Control Systems Technology, IEEE Transactions on*, 11(5):612–628, 2003.

[26] D. Swaroop and J.K. Hedrick. String stability of interconnected systems. *Automatic Control, IEEE Transactions on*, 41(3):349–357, 1996.

[27] M. Šebek. 2-D exact model matching. *IEEE Transactions on Automatic Control*, 28(2):215–217, 1983.

[28] M. Šebek. On 2-D pole placement. *IEEE Transactions on Automatic Control*, 30(8):819–822, 1985.

[29] M. Šebek. n-D polynomial matrix equations. *Automatic Control, IEEE Transactions on*, 33(5):499–502, 1988.

[30] M. Šebek. Polynomial solution of 2D Kalman-Bucy filtering problem. *Automatic Control, IEEE Transactions on*, 37(10):1530–1533, 1992.

[31] M. Šebek. *Multi-Dimensional Systems: Control via Polynomial Techniques*. DrSc dissertation, Czechoslovak Academy of Sciences, 1994.

[32] M. Šebek and F.J. Kraus. Stochastic LQ-optimal control for 2-D systems. *Multidimensional Systems and Signal Processing*, 6(4):275–285, October 1995.

Centrum pro rozvoj výzkumu pokročilých řídicích a senzorických technologií
CZ.1.07/2.3.00/09.0031

Ústav automatizace a měřicí techniky
VUT v Brně
Kolejní 2906/4
612 00 Brno
Česká Republika

http://www.crr.vutbr.cz
info@crr.vutbr.cz