INVESTICE DO ROZVOJE VZDĚLÁVÁNÍ

# Lineární systémy: optimální a prediktivní řízení

Učební texty k semináři


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TENTO STUDIJNÍ MATERIÁL JE SPOLUFINANCOVÁN EVROPSKÝM SOCIÁLNÍM FONDEM A STÁTNÍM ROZPOČTEM ČESKÉ REPUBLIKY

# CONTENTS

# 1. OPTIMAL AND SUBOPTIMAL DECOUPLING CONTROLLERS

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*The problem of decoupling a linear system by dynamic compensation into multi-input multi-output subsystems is studied. The set of all controllers that decouple and stabilize the system is determined in a parametric form. Optimal and suboptimal decoupling controllers are then obtained by an appropriate selection of the parameters.*

## 1.1. Introduction

Consider a linear, time-invariant, differential system that is governed by the input-output relation

$$y = S_y u, \tag{1}$$

where $u$ is the $q$-vector input, $y$ is the $p$-vector output and $S_y$ is the transfer matrix of the system. It is assumed that $S_y$ is a proper real rational matrix.

Let $p_1, ..., p_k$ be given positive integers that satisfy

$$\sum_{i=1}^{k} p_i = p\ .$$

System (1) is said to be decoupled, or more specifically ($p_1, ..., p_k$)-decoupled, if there exist positive integers $q_1, ..., q_k$ satisfying

$$\sum_{i=1}^{k} q_i = q$$

such that $S_y$ has the block diagonal form

$$S_y := \begin{bmatrix} S_1 & & \\ & \ddots & \\ & & S_k \end{bmatrix},$$

where $S_i$ is $p_i \times q_i$.

This is not a generic property of the system, but it can be achieved by a suitable compensation [2], [3], [5]. To this effect, let $z$ denote the $m$-vector output of the system that is available for measurement and let it be related with the input by the equation

$$z = S_z u \,, \tag{2}$$

where $S_z$ is a proper real rational matrix.

The most suitable linear, time-invariant, differential controller can then be described by the equation

$$u = K_v v + K_z z \,, \tag{3}$$

where $v$ is an external reference input of appropriate

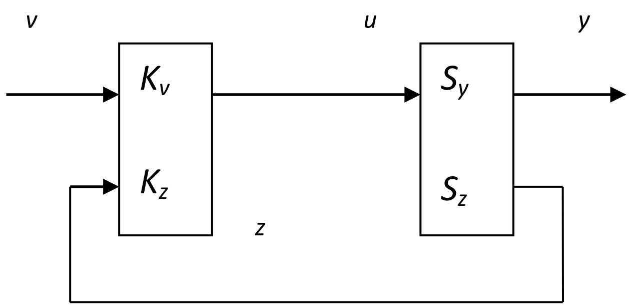


Fig. 1.1. Control system

dimension, say $r$. As it is seen in Fig. 1.1, (3) is a two-degree-of-freedom controller. We assume that both $K_v$ and $K_z$ are proper real rational matrices.

The decoupling problem is then to find matrices $K_v$ and $K_z$ such that the transfer matrix

$$T = S_y (I - K_z S_z)^{-1} K_v$$

from $v$ to $y$ be suitably block diagonal.

Obviously, unless additional provisions are made, the decoupling problem is trivial as it could be solved by $K_v = 0$. Thus it is necessary to impose certain admissibility condition on the decoupling controller in order to make the problem meaningful, for example

$$\operatorname{rank} T = \operatorname{rank} S_y$$

over R($s$), the field of rational functions.

Another requirement, frequently imposed on the decoupled system in practice, is that of stability. This requirement means that the states of the system go to zero from any initial values.

## 1.2. Decoupling and stability

A stable system gives rise to a proper and stable transfer function. In order to study stability of the decoupled system it is convenient to express the transfer matrices in (1), (2), and (3) in the following fractional form

$$\begin{bmatrix} S_z \\ S_y \end{bmatrix} := \begin{bmatrix} B \\ C \end{bmatrix} A^{-1} \tag{6}$$

$$\begin{bmatrix} K_z & K_y \end{bmatrix} := P^{-1} \begin{bmatrix} -Q & R \end{bmatrix}, \tag{7}$$

where

$$A, \begin{bmatrix} B \\ C \end{bmatrix}$$

are proper stable rational matrices that are right coprime and

$$P, \begin{bmatrix} -Q & R \end{bmatrix}$$

are proper and stable rational matrices that are left coprime.

The overall system transfer function then reads

$$T = C(PA + QB)^{-1} R\,. \tag{8}$$

Based on the partition ($p_1$, ..., $p_k$), write

$$C := \begin{bmatrix} C_1 \\ \vdots \\ C_k \end{bmatrix}, \tag{9}$$

where $C_i$ is a $p_i \times q$ submatrix.

We suppose that the given system as well as the controller is a jointly stabilizable and detectable system. Under this assumption, the following solvability condition is proved in [5].

*Theorem 1*. Given system (1), (2) in fractional form (6) and partition (9), there exists an admissible controller (3) such that the overall system is

(i) stable if and only if

A and B are right coprime, (10)

(ii)decoupled if and only if

$$\sum_{i=1}^{k} \text{rank}\, C_i = \text{rank}\, C \quad (11)$$

The interpretation of these solvability conditions is as follows. Condition (10) means that the given system is detectable from the measured output *z*. Condition (11) calls for the linear independence of any two outputs of the given system that belong to different blocks.

### 1.3. Controller parameterization

When a decoupling and stabilizing controller exists, we shall parameterize the class of all such controllers using the Youla-Kučera parameterization [3], [8].

Suppose (10) holds. Let $\overline{P}, \overline{Q}$ be any proper and stable rational matrix solution pair of the equation

$$PA + QB = I \quad (12)$$

Then the solution class of (12) is given by

$$P = \overline{P} + W\overline{B},\ \ Q = \overline{Q} - W\overline{A}, \quad (13)$$

where $\overline{A}$ and $\overline{B}$ are left coprime, proper and stable rational matrices such that

$$\overline{A}^{-1}\overline{B} = BA^{-1} \quad (14)$$

and $W$ is an arbitrary proper and stable rational matrix parameter.

The class of all stabilizing proper rational $K_z$ is then obtained in the form

$$K_z = -P^{-1}Q = -(\overline{P} + W\overline{B})^{-1}(\overline{Q} - W\overline{A}),$$

where the parameter $W$ is constrained so that the inverse of $\bar{P}+W\bar{B}$ exists and is proper rational.

Denote

$$r_i := \operatorname{rank} C_i, \quad i = 1, \ldots, k.$$

Let $U_i$ be a $p_i \times p_i$ unimodular proper and stable rational

matrix such that

$$C_i = U_i \begin{bmatrix} C_i' \\ 0 \end{bmatrix},$$

where the rows of $C_i'$ are linearly independent over R($s$). If (11) holds, then

$$C' := \begin{bmatrix} C_1' \\ \vdots \\ C_k' \end{bmatrix}$$

have linearly independent rows over R($s$). Let $U'$ be a $q \times q$ unimodular proper and stable rational matrix such that

$$C'U' := \begin{bmatrix} D_1 & & & 0 \\ & \ddots & & \vdots \\ & & D_k & 0 \end{bmatrix}, \tag{15}$$

where $D_i$ is a $r_i \times r_i$ diagonal, proper stable rational matrix.

Partition the $q \times q$ unimodular matrix $U'$ defined in (15) as

$$U' = \begin{bmatrix} U_r' & U_{q-r}' \end{bmatrix},$$

where $U_r'$ has $r$ columns with $r$ defined by

$$r := \sum_{i=1}^{k} r_i.$$

The class of all decoupling proper rational $K_v$ is then given by $K_v = P^{-1}R$ with $P$ determined in (13) and

$$R = U_r' \begin{bmatrix} V_1 & & \\ & \ddots & \\ & & V_k \end{bmatrix}, \tag{16}$$

where $V_i$ is an arbitrary non-singular $r_i \times r_i$ proper and stable rational matrix parameter. The matrices $V_1$, ..., $V_k$ in turn parameterize the class of achievable block-diagonal transfer matrices (8) as follows

$$T = \begin{bmatrix} U_1 & & \\ & \ddots & \\ & & U_k \end{bmatrix} \begin{bmatrix} \begin{bmatrix} D_1 \\ 0 \end{bmatrix} & & \\ & \ddots & \\ & & \begin{bmatrix} D_k \\ 0 \end{bmatrix} \end{bmatrix} \begin{bmatrix} V_1 & & \\ & \ddots & \\ & & V_k \end{bmatrix} \tag{17}$$

The parameterization of decoupling stabilizing controllers reveals that decoupling and stabilization are two independent issues [1], [5].

## 1.4. Asymptotic tracking

The decoupling constraint can deteriorate system's performance. The bonus of having a parameterized solution set is that the lost performance can easily be controlled by an appropriate choice of the parameters $V_1$, ..., $V_k$ and $W$.

Suppose that the control objective is for each block of outputs $y_i$ to asymptotically track the corresponding block of reference inputs $v_i$. Thus suppose that $p_i = r_i$ for $i = 1, ..., k$, i.e., there are as many reference inputs as controlled outputs in each block. The tracking error for each block is

$$e_i := v_i - y_i = H_i v_i .$$

Suppose that the reference input is given by

$$v_i = G_i^{-1} g_i , \tag{18}$$

where $G_i$ is a fixed proper and stable rational matrix and $g_i$ is an unspecified proper and stable rational vector that captures the effect of initial conditions. Thus (18) defines a class of references with a specified dynamics.

Asymptotic tracking means that

$$e_i = H_i G_i^{-1} g_i$$

is a proper and stable rational vector. Thus $G_i$ must be absorbed in $H_i$ . In view of (17), $H_i$ has the generic form

$$H_i = I - F_i V_i \,,$$

where $F_i := U_i D_i$ and $V_i$ are proper and stable rational matrices with $F_i$ fixed and $V_i$ an arbitrary parameter to be specified. Therefore, asymptotic tracking is possible if and only if there exists a proper and stable rational matrix $Z_i$ satisfying

$$F_i V_i + Z_i G_i = I \,. \tag{19}$$

Let $\bar{V}_i, \bar{Z}_i$ be any solution pair of equation (19). Then the solution class of (19) is given by

$$V_i = \bar{V}_i + N_i G_i \,, \;\; Z_i = \bar{Z}_i - F_i N_i \,,$$

where $N_i$ is an arbitrary proper and stable rational matrix parameter. Thus, the set of reference-to-error transfer functions that achieve asymptotic reference tracking in a decoupled system is

$$H_i = \bar{Z}_i G_i - F_i N_i G_i \,. \tag{20}$$

## 1.5. Optimal controllers

The benefits of controller parameterization will now be demonstrated in the case of $H_2$ control design [5], [6], [7].

Suppose that for each block, the reference-to-error transfer function $H_i$ parameterized in (20) is to have least $H_2$ norm with respect to $N_i$. So as to achieve this task, determine the inner-outer factorization of $F_i$ ,

$$F_i := F_{iI} F_{iO} \,,$$

where $F_{iI}$ is inner and $F_{iO}$ is outer. Note that $G_i$ is outer for typical references such as steps, ramps, or harmonic signals.

As $F_{iI}$ is inner, premultiplication by $F_{iI}^{-1}$ preserves the $H_2$ norm,

$$\|H_i\| = \|F_{iI}^{-1}H_i\| = \|F_{iI}^{-1}\overline{Z}_i G_i - F_{iO}N_i G_i\| . \tag{21}$$

Write

$$F_{iI}^{-1}\overline{Z}_i G_i = F_{iI}^{-1}K_i + L_i ,$$

where $K_i$, $L_i$ are proper and stable rational matrices with $K_i$ strictly proper. Note that $F_{iI}^{-1}$ has poles only in Re$s > 0$. Then

$\|H_i\|^2 = \|F_{iI}^{-1}K_i + (L_i - F_{iO}N_i G_i)\|^2 = \|F_{iI}^{-1}K_i\|^2 + \| L_i - F_{iO}N_i G_i\|^2$ because the cross terms contribute nothing to the norm. This is a complete square in which only the second term depends
on $N_i$ . Therefore, a unique $N_i$ that attains the minimum of the norm for subsystem $i$ is

$$N_i = F_{iO}^{-1}L_i G_i^{-1} \tag{22}$$

provided $N_i$ is proper and stable rational matrix.

## 1.6. Suboptimal controllers

Unfortunately, matrix (22) is generically unstable for typical references due to the presence of $j\omega$-zeros in $G_i$. This impasse can be obviated by sacrificing the optimality and focusing on suboptimal controllers.

Select proper and stable rational matrices $M_i$, $N_i$ so that

$$L_i = M_i + F_{iO}N_i G_i \tag{23}$$

holds with $M_i$ strictly proper and having a small $H_2$ norm; in fact, as small as desired. Then, using (21),

$$\|H_i\|^2 = \|F_{iI}^{-1}K_i\|^2 + \| M_i\|^2 \tag{24}$$

and the parameter $M_i$ defines a suboptimal controller, for which the resulting $H_2$ norm of $H_i$ is only an incremental addition to the unattainable infimum.

## 1.7. Example

Consider a system defined by (1), (2) with transfer matrices

$$S_y = \begin{bmatrix} 1 & \dfrac{s+2}{s-1} \\ \dfrac{s-1}{s+2} & 2 \end{bmatrix}, \quad S_z = \begin{bmatrix} \dfrac{2s+1}{s+2} & \dfrac{3s}{s-1} \\ \dfrac{s-1}{s+2} & 2 \end{bmatrix}.$$

Thus the measurement output $z$ is different from the output $y$ to be decoupled in that it involves a non-unity feedback sensor.

The task is to determine a two-degree-of-freedom controller (3) that (1, 1)-decouples and stabilizes the system.

The first step is to obtain a proper and stable fractional representation (6) for the system. Standard calculations yield

$$A = \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{s-1}{s+2} \end{bmatrix}, B = \begin{bmatrix} \dfrac{2s+1}{s+2} & \dfrac{3s}{s+2} \\ \dfrac{s-1}{s+2} & 2\dfrac{s-1}{s+2} \end{bmatrix}, C = \begin{bmatrix} \dfrac{1}{s+2} & \dfrac{1}{s+2} \\ \dfrac{s-1}{s+2} & 2\dfrac{s-1}{s+2} \end{bmatrix}.$$

Now apply Theorem 1. Since (10) holds, a stabilizing controller exists. Since (11) holds as well, also an admissible decoupling controller exists.

All stabilizing and decoupling controllers will be parameterized using the fractional representation (7). To obtain the feedback part of the controller, we consider any particular solution of equation (12), for example

$$\overline{P} = \begin{bmatrix} 1 & 0 \\ -\dfrac{2s+1}{s+2} & -2 \end{bmatrix}, \quad \overline{Q} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

The solution class (13) of equation (12) is

$$P = \begin{bmatrix} 1 & 0 \\ -\dfrac{2s+1}{s+2} & -2 \end{bmatrix} + W \begin{bmatrix} \dfrac{s-1}{s+2} & 2 \\ -\dfrac{(s-1)(2s+1)}{(s+2)^2} & -\dfrac{3s}{s+2} \end{bmatrix}, \tag{25}$$

$$Q = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - W \begin{bmatrix} 0 & 1 \\ -\dfrac{s-1}{s+2} & 0 \end{bmatrix}, \tag{26}$$

where use has been made of (14).

To obtain the feedforward part of the controller, note that $U_1 = U_2 = 1$ and the unimodular matrix defined in (15) is

$$U' = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}.$$

Thus (16) yields

$$R = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix}. \tag{27}$$

The matrices $P$, $Q$ in (25), (26) and $R$ in (27) define the class of all controllers that solve the decoupling problem. The parameters $V_1$, $V_2$ are free non-zero proper and stable rational functions and $W$ is permitted to range over proper and stable rational $2 \times 2$ matrices so that the inverse of $P$ exists and is proper.

The decoupled transfer matrices that can be achieved in this example are given by (17) as

$$T = \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{s-1}{s+2} \end{bmatrix} \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix}.$$

Suppose that the decoupled outputs are to follow any step references and that the transients are to be optimized in terms of the $H_2$ norm. Thus, put

$$G_1 = G_2 = \frac{s}{s+1}$$

and solve equation (19) channel by channel. Clearly $V_1 = 1$, $Z_1 = 0$ is an optimal solution that yields $H_1 = 0$. On the other hand,

$$V_2 = \frac{s-2}{s+1} + N_2 \frac{s}{s+1}, \quad Z_2 = \frac{6}{s+2} - \frac{s-1}{s+2} N_2$$

and the inner-outer factorization of

$$F_2 = \frac{s-1}{s+2}$$

is seen to be

$$F_{2I} = \frac{s-1}{s+1}, \quad F_{2O} = \frac{s+1}{s+2}.$$

Then

$$F_{iI}^{-1}\overline{Z}_2 G_2 = \frac{s+1}{s-1}\frac{6}{s+2}\frac{s}{s+1} = \frac{s+1}{s-1}\frac{2}{s+1} + \frac{4}{s+2}$$

so that

$$K_2 = \frac{2}{s+1}, \; L_2 = \frac{4}{s+2}.$$

Thus, from (22),

$$N_2 = \frac{6}{s}$$

and the infimum of $\|H_2\|$ cannot be attained.

To obtain a suboptimal controller, choose

$$M_2 = \frac{2\varepsilon}{s+\varepsilon}, \; N_2 = \frac{4-2\varepsilon}{s+\varepsilon}$$

with ε > 0 arbitrarily small, in order to satisfy (22). Then

$$V_2 = \frac{s+2}{s+1}\frac{s-\varepsilon}{s+\varepsilon}, \; Z_2 = \frac{2+2\varepsilon}{s+\varepsilon}, \; H_2 = \frac{2+2\varepsilon}{s+\varepsilon}\frac{s}{s+1}$$

and it follows from (24) that

$$\|H_2\|^2 = \left\|\frac{2}{s+1}\right\|^2 + \left\|\frac{2\varepsilon}{s+\varepsilon}\right\|^2 = 2+2\varepsilon$$

is arbitrarily close to the infimum value of 2

It follows from (27) that a suboptimal $R$ is

$$R = \begin{bmatrix} 2 & -\dfrac{s+2}{s+1}\dfrac{s-\varepsilon}{s+\varepsilon} \\ -1 & \dfrac{s+2}{s+1}\dfrac{s-\varepsilon}{s+\varepsilon} \end{bmatrix}$$

and the overall system has the transfer function

$$T = \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{s-1}{s+1}\dfrac{s-\varepsilon}{s+\varepsilon} \end{bmatrix}.$$

## 1.8. References


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# 2. MODELOVÁNÍ A FYZIKA BUDOV

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*V této kapitole se seznámíme s problematikou termodynamiky budov, ukážeme si základní fyzikální principy, na kterých budovy fungují, a připravíme si východiska pro identifikaci budov.*

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## 2.1. Motivace

Podle dostupných zdrojů spotřebují budovy celosvětově 20-40 % veškeré energie, navíc v rozvinutých zemích tento podíl stoupá o 0,5-5 % ročně. Například v USA v roce 2010 spotřebovaly budovy 41 % primární energie (průmysl 31 % a doprava 28 %). Přestože se efektivita používaných systémů vytápění, ventilace a klimatizace v budovách v posledních letech výrazně zlepšila, stále je zde prostor pro úspory v oblasti algoritmizace řízení, jak si v této kapitole ukážeme.

Se svojí relativně vysokou spotřebou jsou tedy budovy vhodným objektem k úsporám energií. V současné době se úspory energií v budovách řeší řadou různých způsobů, mezi něž patří například:

- Zateplení fasády
- Dobrá okna
- Snížení vnitřní teploty
- Alternativní zdroje energie
- Lepší regulace

Je zřejmé, že zatímco prvně uvedené možnosti opravdu snižují energetickou náročnost budovy, ovšem za cenu vysokých vstupních nákladů a dlouhé návratnosti, lepší regulace „pouze“ lépe využívá energetické možnosti dané stávajícím technickým stavem budovy. **Lepší regulace** tedy nešetří v pravém slova smyslu, ale **zabraňuje plýtvání**.

V současnosti se pro regulaci vytápění (ale i chlazení a klimatizace) používají v zásadě následující metody:

- Termostat
  Jedná se o asi nejstarší regulaci vůbec. Už pravěký člověk si prostě přiložil na oheň, když mu byla zima – moderní nástěnné termostaty s hysterezí, případně učícími se algoritmy, jsou pouze pokračovatelem této tradice. Nicméně termostat funguje velmi spolehlivě, zachovává zpětnou vazbu od místnosti, ale nedokáže reagovat na rychlé, dynamické děje dané například prudkou změnou počasí.

- Ekvitermní regulace
  Na základě venkovní teploty nastavuje ekvitermní regulace teplotu topné vody, která pak jde do jednotlivých místností – jedná se tedy o přímovazební regulátor. Ekvitermní regulace je velmi robustní, ale postrádá zpětnou vazbu od místností a není dynamická.

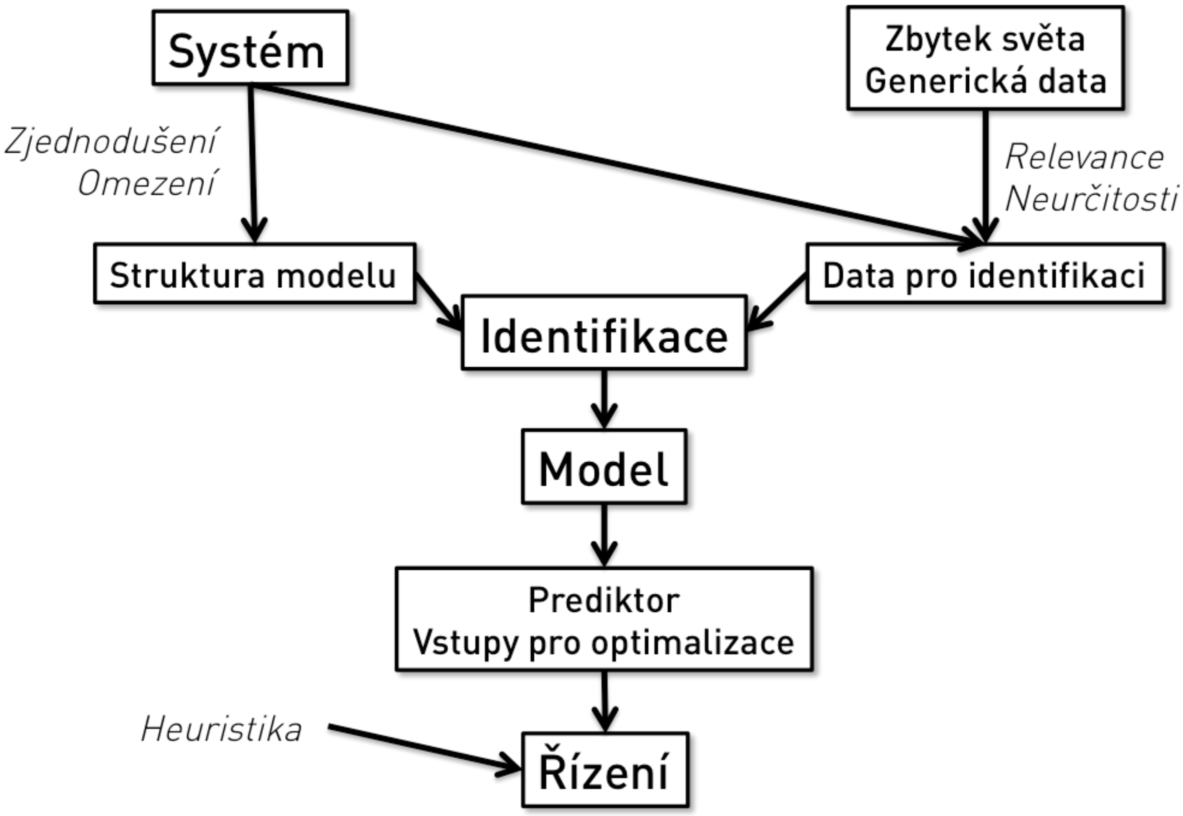
- Regulace podle referenční místnosti (např. PID)
  Zpětnovazební, dynamický regulátor řídí teplotu topné vody podle teploty v referenční místnosti. Postrádá však informace o počasí a je citlivý na lokální poruchy (např. na otevřené okno).

- Podmínkové řízení – Rule Based Control (RBC)
  Řízení typu if-then-else je velmi rozšířené zvláště u velkých budov, protože dokáže zkombinovat údaje z mnoha senzorů a budovu řídit komplexně. Návrh takové regulace je nicméně velmi složitý a vyžaduje velkou inženýrskou zkušenost.

- MPC
  V posledních několika letech se experimentuje s prediktivním regulátorem, který by byl vhodný pro velké budovy, byl by dynamický, robustní a bral v úvahu předpovědi počasí.

Obecně řečeno mám dvě možnosti – buď zvolím klasickou metody regulace, nebo se rozhodnu pro sofistikovanou regulaci komplexního systému. V tom případě však musím vědět, že se mi nákladná investice do nového řídicího systému s dosaženými úsporami vrátí. My budeme předpokládat, že ano, a v dalším textu ukážeme, jak moderní regulaci budovy navrhnout.

## 2.2. Koncepce regulace




**Obrázek 2.1 Nějaký obrázek, s číslováním podle kapitol**

Na obrázku výše je schéma postupu tvorby regulace. Nejprve si musím vymezit svůj systém a oddělit jej od zbytku světa. Na základě své znalosti si pak zvolím strukturu modelu, společně s naměřenými daty potom identifikuji neznámé parametry a získám skutečný model. Ten potom použiji jako prediktor, resp. vstup pro optimalizaci výsledného regulátoru.

Z hlediska řízení tedy nejprve najdu model systému

$$y = P(u, x, t) \tag{2.1}$$

a potom hledám takové optimální vstupy do systému, které splňují optimalizační požadavek

$$u_{optimal} = \arg\min_{\mathrm{u}} J(P(u, x, t), u, t) \tag{2.2}$$

## 2.3. Budova jako systém

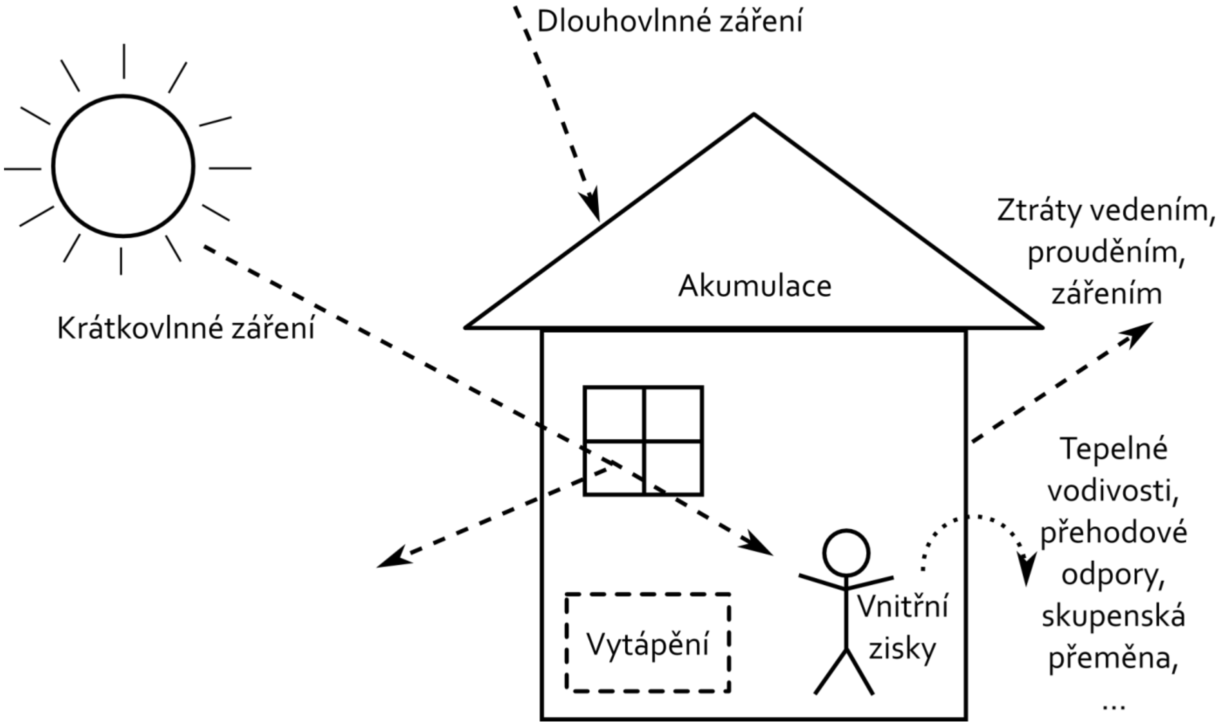
V budově uvažujeme typicky následující fyzikální děje:

- Přenos tepla
- Tepelná rovnováha
- Tepelná pohoda
- Energetická spotřeba
- Okrajové podmínky

Tyto děje si nyní ve stručnosti představíme.

### 2.3.1. *Přenos tepla*

Teplo se šíří vedením, prouděním a zářením. Na následujícím obrázku je znázorněno, jak situace přenosů tepla v budově vypadá.




Obrázek 2.2 Tepelné přenosy v budově

Je zřejmé, že obrázek je velmi zjednodušený – přenosů v budově je celá řada, neboť dochází k různým odrazům, ztrátám, akumulacím atd.

### 2.3.2. *Tepelná rovnováha*

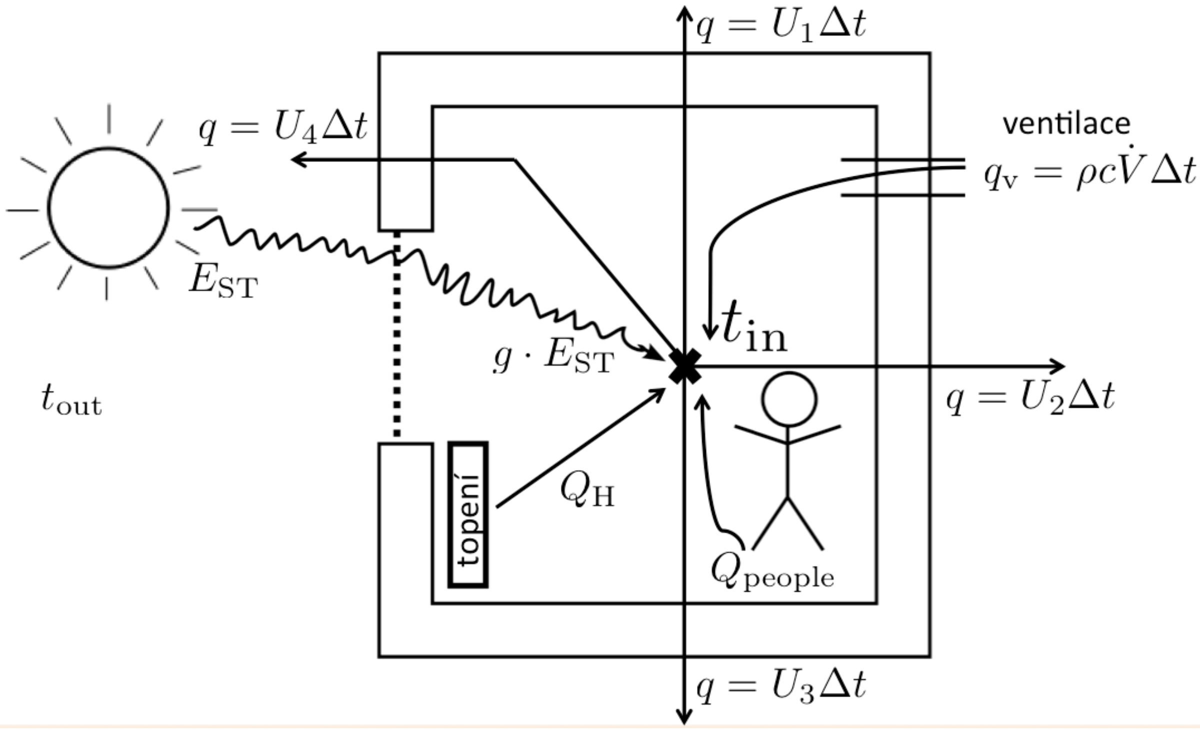
Tepelnou rovnováhou budovy rozumíme v ustáleném stavu stav popsaný rovnicí

$$\sum Q = 0 \tag{2.3}$$

resp. v dynamickém tvaru

$$\sum Q = C\,\frac{dt}{d\tau} \tag{2.4}$$

kde $\tau$ je čas a $t$ je teplota. Situaci si opět můžeme ukázat na následujím obrázku.




Obrázek 2.3 Tepelná rovnováha místnosti

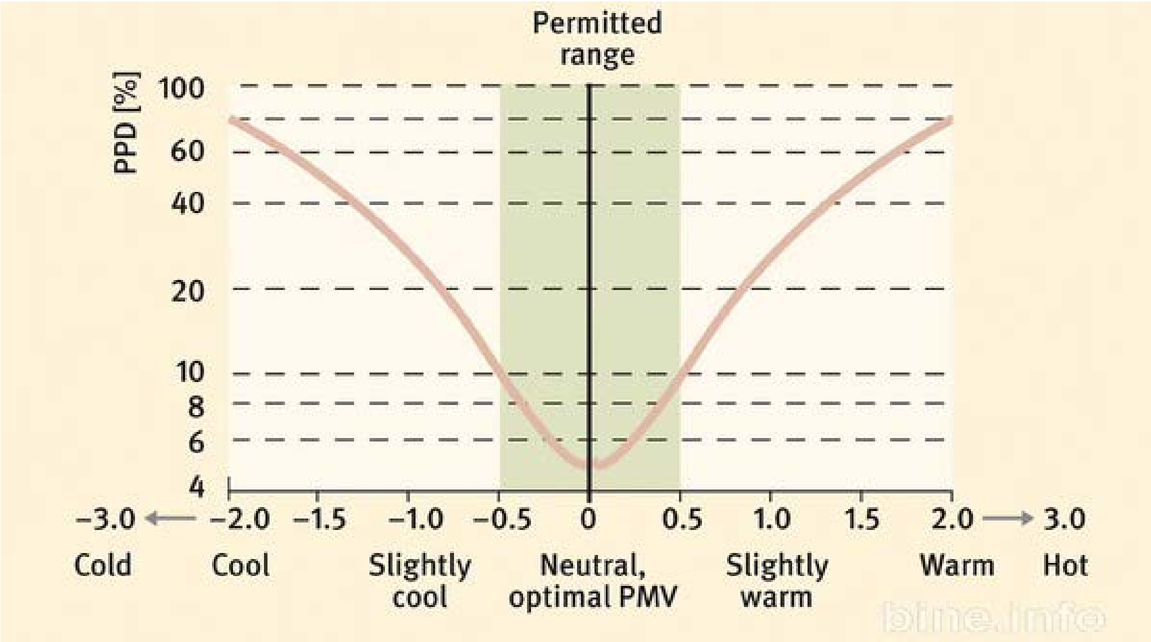
Pro tuto situaci musí platit:

$$g.E_{ST} + Q_{people} + Q_V + Q_H = \sum_i U_i A_i \Delta t \qquad (2.5)$$

tedy zisky ze slunce, vnitřní zisky, ventilační teplo a teplo ze systému vytápění se musí rovnat ztrátám skrz zdi.

### 2.3.3. *Tepelná pohoda*

V současnosti je velmi populární tzv. PMV (Predicted Mean Vote). Vychází se z toho, že tepelná pohoda člověka závisí na několika různých vlivech – na teplotě vzduchu, teplotě záření okolních předmětů, relativní vlhkosti vzduchu, rychlosti proudění vzduchu v místnosti, ale také na rychlosti metabolismu člověka a na jeho oblečení. Pro numerický výpočet PMV se používají velmi sofistikované rovnice vycházející z empirických měření na statisticky významném vzorku populace, nicméně princip PMV je jednoduchý a lze jej ilustrovat následujím obrázkem.




**Obrázek 2.4 Predicted Mean Vote**

Zkratka PPD znamená Predicted Percentage of Dissatisfaction, tj. podíl lidí, kteří budou v budově nespokojeni s tepelnou pohodou. Je zřejmé, že i když budova bude odpovídat kategorii PMV=0, bude uvnitř stále cca 5 % lidí nespokojených. Z hlediska praxe je důležité, aby PMV leželo v intervalu <-0,5; 0,5>.

## 2.3.4. *Energetická spotřeba*

Energetickou spotřebou se z hlediska termodynamiky rozumí teplo dodané do budovy, zatímco jako uživatele nás zajímá spíše teplo účtované dodavatelem, ovšem může náš zajímat i primární energie. Situaci si můžeme znázornit následující tabulkou:

| | |
|---|---|
| Teplo dodané budově | |
| | účinnost předání tepla |
| Teplo dodané do otopného systému | |
| | ztráty na vedení |
| | účinnost řídicího systému |
| Teplo dodané budově | |
| | účinnost předávky tepla |
| | účinnost pomocných zařízení (čerpadla, ventily, ...) |
| Energie dodaná budově | |
| | ztráty distribuční sítě |
| Vyrobená energie | |
| | účinnost výroby |
| Primární energie | |

Okrajové podmínky

Mezi okrajové podmínky patří například:

- Venkovní teplota
- Sluneční svit
- Rozptýlená dlouhovlnná radiace
- Vítr (síla, směr)
- Vlhkost vzduchu
- Srážky
- Zastínění (stromy, okolními budovami)
- Orientace (S-J-V-Z)
- Geologické podloží
- atd.

## 2.4. Přechod k modelování budovy

Na základě znalostí o budově a její fyzice mohu nyní přistoupit k návrhu koncepce modelu. Model bude sloužit jako součást MPC regulátoru a s ohledem na tuto skutečnost jej budu navrhovat.

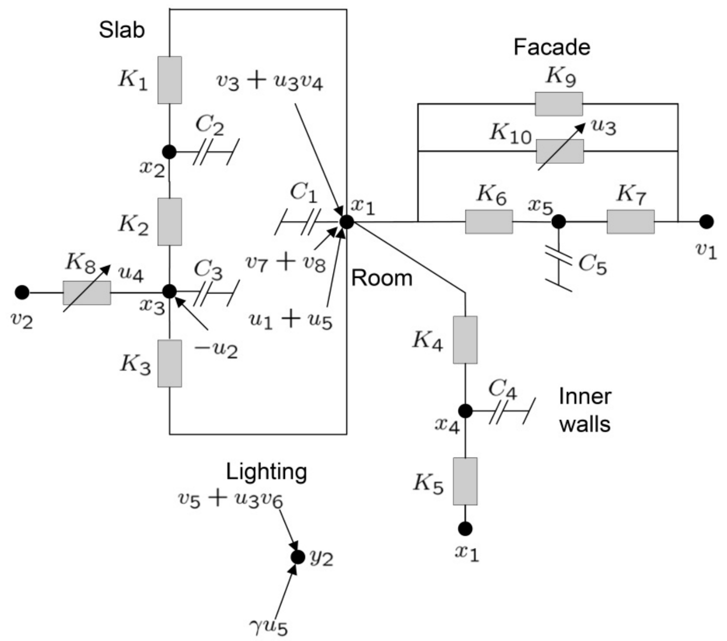
Budu tedy hledat lineární, časově invariantní, stochastický model ve tvaru

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) + w(t) \\ y(t) &= Cx(t) + Du(t) + v(t) \end{aligned} \tag{2.6}$$

Jak jsme viděli v předchozí části, fyzikální principy, na kterých budovy fungují, jsou velmi složité a budovu nelze modelovat přísně deterministicky. Zatímco do deterministické části modelu se budu snažit začlenit hlavně zdroje tepla, ztráty, dynamiku stavby, její akumulaci a některé okrajové podmínky (hlavně vliv slunečního záření a okolní teploty), do stochastické části ponechám fenomény jako kolísání obsazenosti budovy, náhodné děje typu otevírání oken a dveří, vnitřní regulační smyčky, individuální geometrii místností apod.

Ke struktuře modelu mohu přistoupit třemi různými způsoby.

Buď zvolím **white-box model**, tj. napíšu si všechny rovnice, které budovu popisují, a identifikuji jednotlivé parametry. Tento přístup je velmi komplikovaný a pro modely pro řízení se v případě budov nepoužívá. Další možností je použít **gray-box model**, v našem případě například ekvivalentní RC síť a identifikovat její fiktivní parametry („kapacitu“ a „odpor“). Tento přístup může být velmi úspěšný, ale není předmětem tohoto kurzu. Poslední možností je **black-box model**, tj. identifikujeme pomocí statistických identifikací parametry systémových matic.




**Obrázek 2.5 Gray-box model – ekvivalentní RC síť**

Zvláštním případem identifikace white-box modelu je subspace identifikace, která bude tématem následující kapitoly.

Poslední, co nám ke spuštění identifikace chybí, jsou identifikační data. Ta lze získat dvojím způsobem. Buď mohu použít data, která jsou měřená přímo na budově. Data pro identifikaci však musí být dostatečně pestrá a majitel budovy nemusí souhlasit s prováděním „identifikačních experimentů“. Proto je alternativou namodelovat budovy ve specializovaném SW (TRNSYS, Energy Plus apod.). Je však třeba upozornit, že tyto softwary vznikly z požadavku stavebních inženýrů a nejsou primárně určeny pro tvorbu modelů vhodných k regulaci. Je možné je použít jako podpůrný nástroj, z větší části však zůstaneme závislí na reálných datech.

## 2.5. Shrnutí kapitoly

Závěry této kapitoly lze v podstatě shrnout do následujících bodů:

- Chceme ušetřit energii v budovách pomocí pokročilé regulace
- Regulace bude založená na modelu budovy
- Jako systém je budova velmi komplexní
- Její model bude mít deterministickou a stochastickou část
- Model může mít gray-box nebo black-box strukturu
- My se v dalších kapitolách zaměříme na subspace identifikace black-box struktury modelu

Kapitola 3

# Subspace Identification

Ing. Lukáš Ferkl, Ph.D.

*The family of methods called Subspace State Space Systems IDentification (4SID) are used for the identification of Linear Time Invariant (LTI) state space models directly from the input-output data. They are an alternative to the famous Prediction Error Methods (PEM).*
*Subspace identification algorithms are based on concepts from system theory, linear algebra and statistics. As the most of the real-life system are multiple input multiple output (MIMO) system and as 4SID methods provide state space model of the system, which is probably the most natural expression of the MIMO systems, the 4SID methods appear to be very suitable candidate for identification of MIMO systems.*

## 3.1 Motivation, Introduction

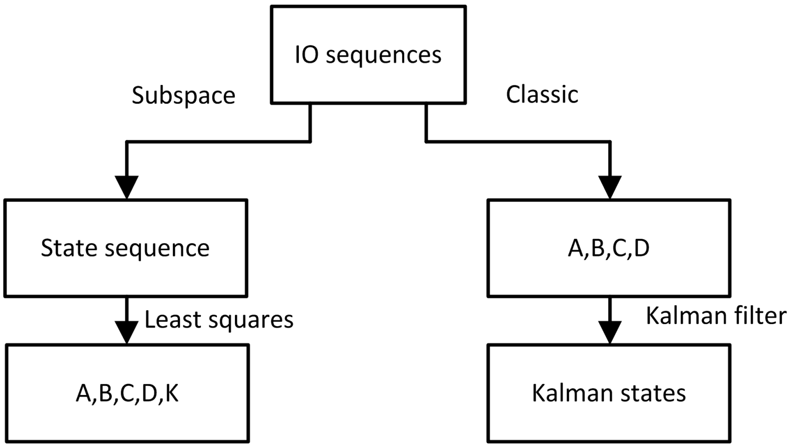
Why should anybody use subspace identification? What is it good for? Is it yet another identification method? Why is it so popular recently? What do you learn, when you complete this lecture? We will try to answer these and other questions in the following.
Well, every control engineer knows, that before any actions taken for control itself, much has to be done. In some methods less, e.g. many "classical" control concepts such as PID come out from the control of the error (as a difference between requested and actual values of the manipulated variable), thus the model of the process is not necessary; on the other hand there are some modern approaches which heavily depend on the good model of the system, such as predictive control. Therefore we should realize, that a good dynamic model at our disposal is oftentimes absolutely crucial for the following control. And as the system to control become more and more complex, we need a suitable identification method to handle these ever more difficult to handle systems. One such a method, or better said a family of methods, which are

capable of handling the multiple input multiple output (MIMO) systems is a family of subspace state-space system identification (4SID).
In the following we will try to provide you with information, that should help you with understanding how the 4SID works, their pros and cons, etc.
First, let us enumerate some characteristics of the subspace identification:

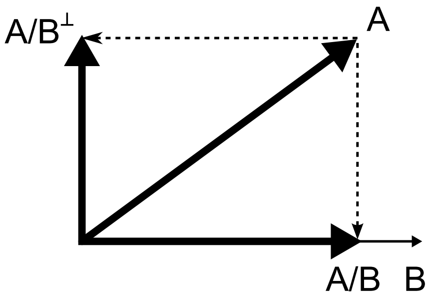
- ✓ numerical algorithm resulting to the estimate of the state space model using input-output data
- ✓ "subspace" expresses the way of obtaining the state sequence: using mathematical tools the state sequence can be recovered from the subspace of some matrices constructed only from input-output data
- ✓ minimum required number of parameters: uses does not need to estimate the system structure (black-box model). The only parameters to set are size of the block Hankel matrix, and order of the system - which estimate is provided within the 4SID identification procedure
- ✓ No iterations, which means that there are no problems with convergence or finding the global optimum.
- ✓ Numerically robust mathematical tools such as SVD or QR
- ✗ Basic version can not handle cases, where the input is correlated with output (e.g. closed loop measurements). In these cases the estimate is heavily biased.
- ✗ Difficult recursive implementation

## 3.2 Mathematical Tools

Before we start with the description of the tools utilized in 4SID, let us explain the basic difference in the classical and subspace identification concepts. Let us have a look at Fig. 3.1. In the classical identification concept one always tries to estimate system matrices directly, or most often intermediary via estimation of some coefficients of the chosen polynomials or more complex functions. Only then, the system states are estimated using Kalman filtering. In sharp contrast, the 4SID methods forms some matrices using only measured input-output data, and then, utilizing concepts of algebra and geometry a state sequence is estimated. Then a problem of classical least squares is formulated wherein the system matrices stand for unknown estimated parameters.




Obrázek 3.1: Comparison between classical and subspace identification methods




Obrázek 3.2: Orthogonal projection principle

Let us now turn to the promised mathematical tools. We will explain the following

- Orthogonal projection of the row space of the matrix $A$ onto the row space of the matrix $B$ is denoted as $A/B$, which is equal to the following

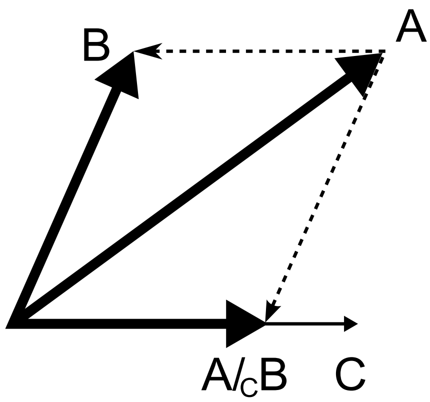
$$A/B = A\Pi_B = AB^T(BB^T)^\dagger B, \tag{3.1}$$

  where † is a Moore-Penrose pseudo-inverse. This means, that the result lies in the row space of B and has the same number or row vectors as did A. Then we have an orthogonal projection onto the complement of the row space of the matrix B denoted as

$$A/B^\perp = A\Pi_B^\perp = I - \Pi_B \tag{3.2}$$

  The orthogonal projection is demonstrated at Fig. 3.2, or a a short animation available at `http://www.youtube.com/watch?v=0ABUhoDzMxU&NR=1`. (3.1) and (3.2) mean, that $A$ can be decomposed as a sum of linear combinations of the rows of $B$ and $B^\perp$.

- Oblique projection has a bit different interpretation. Instead of decomposing $A$ as linear combination of two orthogonal matrices, it can be



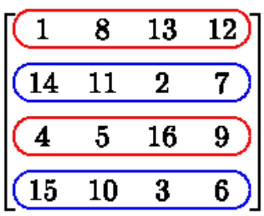

Obrázek 3.3: Oblique projection principle

decomposed as a linear combination of non-orthogonal matrices $B$ and $C$ and of the orthogonal complement of $B$ and $C$, which can be written as

$$A = A\underset{C}{/}B + A\underset{B}{/}C + A/\begin{pmatrix} B \\ C \end{pmatrix}^{\perp}. \tag{3.3}$$

For better understanding, please refer to Fig. 3.3.

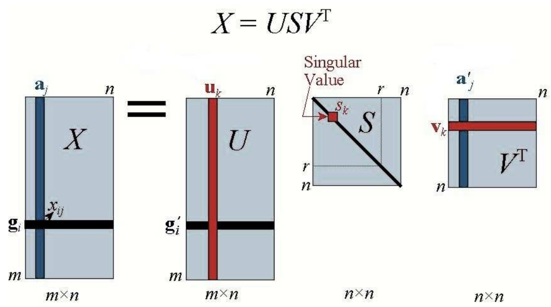
- Matrix row space is the set of all possible linear combinations of its row vectors. The dimension of the row space is called the rank of the matrix. The idea is depicted in Fig. 3.4. Rephrasing the statement in words of math, when $A$ is $m \times n$ matrix, with $r_1 \dots r_m$ rows, the set of all possible linear combinations, i.e. $c_1 r_1 + c_2 r_2 + \dots c_m r_m$ is called a row space of the matrix.




Obrázek 3.4: Matrix row space [1]

- Hankel matrix is a square matrix with constant skew-diagonals. In the context of 4SID algorithm, where the Hankel matrix is constructed using input-output data, the interpretation is quite straightforward. Each column is only a time shift of the previous one.

- SVD decomposition or the singular value decomposition is a factorization of a real or complex matrix, where a $m \times n$ real or complex matrix $X$ is decomposed intro three matrices as depicted in Fig. 3.5. $U$is an $m \times m$ real or complex unitary matrix, $S$ is an $m \times n$ diagonal matrix with nonnegative real numbers on the diagonal (singular values), and $V^*$ (the conjugate transpose of V) is an $n \times n$ real or complex unitary matrix. The $m$ columns of $U$ and the $n$ columns of $V$ are called the left and right singular vectors of $X$, respectively. The SVD is utilized in solving of many such as pseudoinverse, least squares, determining the rank, etc.

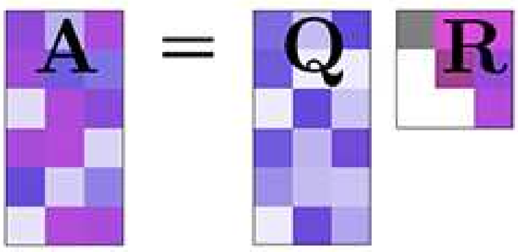



Obrázek 3.5: Principle of SVD

- QR decomposition is a decomposition of a matrix $A$ into a product $A = QR$, where matrix $Q$ is orthogonal and matrix $R$ is an upper triangular. One of the most often problems solved by QR decomposition is linear least squares (LS)problem. If $A$ has $n$ linearly independent columns, then the first $n$ columns of $Q$ form an orthonormal basis for the column space of $A$. In general we can write

$$A = QR = Q \begin{pmatrix} R_1 \\ 0 \end{pmatrix} = (Q_1 \; Q_2) \begin{pmatrix} R_1 \\ 0 \end{pmatrix} = Q_1 R_1, \tag{3.4}$$

with $R_1$ being upper triangular $n \times n$ matrix, $Q_1$ being $m \times n$ (same as the original size of $A$).




Obrázek 3.6: Principle of QR

## 3.3 Identification Procedure

### 3.3.1 Problem Statement

The objective of the subspace algorithm is to find a linear, time invariant, discrete time model in an innovative form

$$
\begin{aligned}
x(k+1) &= Ax(k) + Bu(k) + Ke(k) \\
y(k) &= Cx(k) + Du(k) + e(k),
\end{aligned} \tag{3.5}
$$

based on given measurements of the input $u(k) \in \mathbb{R}^m$ and the output $y(k) \in \mathbb{R}^l$ generated by an unknown stochastic system of order $n$, which is equivalent to the well-known stochastic model as defined in e.g. [3, 2]. Loosely speaking, the objective of the algorithm is to determine the system order $n$ and to find the matrices $A$, $B$, $C$, $D$ and $K$.

### 3.3.2 Algorithm

The entry point to the algorithm are input-output equations as follows:

$$
\begin{aligned}
Y_p &= \Gamma_i X_p^d + H_i^d U_p + Y_p^s \\
Y_f &= \Gamma_i X_f^d + H_i^d U_f + Y_f^s \\
X_f^d &= A^i X_p^d + \Delta_i^d U_p,
\end{aligned} \tag{3.6}
$$

where $Y_p$ and $Y_f$ are the Hankel matrices of past and future outputs, $U_p$ and $U_f$ are the Hankel matrices of past and future inputs, $X_p^d$ and $X_f^d$ are the deterministic Kalman state sequences, $Y_p^s$ and $Y_f^s$ are the stochastic Hankel matrices of past and future outputs, $H_i^d$ is the lower block triangular Toeplitz matrix for the deterministic subsystem (which contains system matrices), $\Gamma_i$ is the extended system observability matrix (which contains matrices $A$ and $C$) and $\Delta_i^d$ is the deterministic reversed extended controllability matrix (which contains matrices $A$ and $B$). Detailed construction of state matrices can is provided later in the text. It is quite straightforward that following holds:

$$
\begin{bmatrix} \hat{X}_{i+1} \\ Y_{i|i} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \hat{X}_i \\ U_{i|i} \end{bmatrix} + \begin{bmatrix} \rho_w \\ \rho_v \end{bmatrix} \tag{3.7}
$$

$\hat{X}_i$ denotes estimate of state sequence, $\rho_w$ and $\rho_v$ are Kalman filter residuals. State sequence estimates are determined as follows:

$$
\begin{aligned}
\hat{X}_i &= \Gamma_i^\dagger \left[\mathfrak{Z}_i - H_i^d U_f\right] \\
\hat{X}_{i+1} &= \Gamma_{i-1}^\dagger \left[\mathfrak{Z}_{i+1} - H_{i+1}^d U_f^-\right],
\end{aligned} \tag{3.8}
$$

with $\mathfrak{Z}_i$ and $\mathfrak{Z}_{i+1}$ defined as oblique projections (see e.g. [4])

$$\begin{aligned} \mathfrak{Z}_i &= Y_f \underset{U_f}{/} W_p \\ \mathfrak{Z}_{i+1} &= Y_f^- \underset{U_f^-}{/} W_p^+, \end{aligned} \tag{3.9}$$

where $W_p$ is a lumped data matrix containing $U_p$ and $Y_p$. Solving (3.7) using least squares methods, we get the state space system description of the system, namely the system in the innovation form (3.5).
Finally, given the estimates of the system matrices $A$, $B$, $C$, $D$ the Kalman gain matrix $K$ can be computed. If an estimate of a state sequence $X$ is known, the problem can be solved by computing the Algebraic Riccati Equation (ARE) in which the covariance matrices are determined from the residuals as follows:

$$\begin{bmatrix} W \\ V \end{bmatrix} = \begin{bmatrix} X_{k+1} \\ Y_k \end{bmatrix} - \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} \begin{bmatrix} X_k \\ U_k \end{bmatrix}, \tag{3.10}$$

where

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} = \frac{1}{N} \left( \begin{bmatrix} W \\ V \end{bmatrix} \begin{bmatrix} W^T & V^T \end{bmatrix} \right). \tag{3.11}$$

### 3.3.3 Matrices Used in Subspace Algorithm

Notation and building-up of the matrices as follows further on were adopted as in [4]. Upper index $d$ and $s$ denotes deterministic and stochastic subsystems, respectively.

#### Data Matrices

Input block Hankel matrix is built-up from input data as follows:

$$U_{0|2i-1} = \left( \begin{array}{ccccc} u_0 & u_1 & u_2 & \cdots & u_{j-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_{i-1} & u_i & u_{i+1} & \cdots & u_{i+j-2} \\ \hline u_i & u_{i+1} & u_{i+2} & \cdots & u_{i+j-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_{2i-1} & u_{2i} & u_{2i+1} & \cdots & u_{2i+j-2} \end{array} \right). \tag{3.12}$$

Input and output Hankel matrices can be grouped as follows:

$$W_p = \left( \frac{U_p}{Y_p} \right) \quad , \quad W_p^+ = \left( \frac{U_p^+}{Y_p^+} \right), \tag{3.13}$$

where $U_{0|i-1} = U_p$ and $U_{i|2i-1} = U_f$ with $U_p$ and $U_f$ denoting the past and future inputs, respectively. The same logic holds for outputs $y(k)$ and noise $e(k)$. Change of indices results in $U_{0|i} = U_p^+$ and $U_{i+1|2i-1} = U_f^-$, respectively.

### System Related Matrices

Extended ($i > n$) observability ($\Gamma_i$) and reversed extended controllability ($\Delta_i$) matrices for deterministic and stochastic subsystems, respectively are defined as follows:

$$\Gamma_i = \begin{pmatrix} C^T & (CA)^T & \dots & (CA^{i-1})^T \end{pmatrix}^T \tag{3.14}$$

$$\Delta_i^d = \begin{pmatrix} A^{i-1}B & A^{i-2}B & \dots & AB & B \end{pmatrix} \tag{3.15}$$

$$\Delta_i^s = \begin{pmatrix} A^{i-1}K & A^{i-2}K & \dots & AK & K \end{pmatrix} \tag{3.16}$$

The lower block triangular Toeplitz matrix for deterministic and stochastic subsystem, respectively are defined as

$$H_i^d = \begin{pmatrix} D & 0 & \dots & 0 \\ CB & D & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{i-2}B & CA^{i-3}B & \dots & D \end{pmatrix}, \tag{3.17}$$

$$H_i^s = \begin{pmatrix} I & 0 & \dots & 0 \\ CK & I & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{i-2}K & CA^{i-3}K & \dots & I \end{pmatrix}. \tag{3.18}$$

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Chapter 4

# Predictive Control for Buildings

Ing. Jiří Cigler

## 4.1 Introduction

### 4.1.1 Motivation

In recent years, there has been a growing concern to achieve energy savings. This has been demonstrated by the governments of many developed countries. For instance, the European Union (EU) presented targets concerning energy cuts defining goals until 2020 [47]: *i)* Reduction in EU greenhouse gas emissions at least 20 % below the 1990 levels, *ii)* 20 % of EU energy consumption to come from renewable resources, *iii)* 20 % reduction in primary energy use compared to projected levels to be achieved by improving energy efficiency. The similar goals, in some cases even more restrictive, have been stated by the U.S. government with minor differences on the level of each state [6].

As the buildings account for about 40 % of total final energy consumption [33] and more than half is consumed in HVAC (Heating, Ventilation and Air Conditioning) systems, an efficient building climate control can significantly contribute to reduction of the power demands and lower thus the greenhouse gas emissions.

In addition, for instance in the U.S., there are about one to 2 million buildings being newly constructed every year. However, there are approximately 110 million existing buildings consuming much more energy *per se* than new buildings constructed according to current standards. Even when each of the new buildings would use net-zero-energy technology, it will take a long time to achieve significant difference on the overall energy bill [37]. Therefore, a much more productive approach for achieving the strict energy cuts would be to focus on the retrofitting of the existing buildings or by improvements of Building Automation Systems (BAS) and their algorithms that can be achieved with minimal additional cost. In this paper, we restrict ourselves only to improvements in BAS algorithms. The effort to implement advanced control algorithms in buildings has been shown by the activity of the leading academic and industrial teams in the area of HVAC control [8, 21, 43, 22, 11].

### 4.1.2 State-of-the-art in advanced control of HVAC systems

Recently, there have emerged two main research trends in the field of advanced HVAC control *i)* learning based approaches like artificial inteligence; neural networks; fuzzy and adaptive fuzzy neural networks etc. *ii)* Model based Predictive Control (MPC) techniques that stand on the principles of the classical control.

The approaches from the former group are used in HVAC systems for their capability in dealing with nonlinearities as well as their capabilities to handle Multi-Input Multi-Output (MIMO) systems. These approaches can be for instance used to cut down the time needed for tuning the supervisory controller [45], to control cooling system with several types of cooling strategies [42] or to optimize occupants' thermal comfort making use of ventilation control [11].

The latter technique can handle MIMO systems from its very nature and usually relies on the physically based mathematical model of the HVAC system and building dynamics. The aim of MPC is to design control inputs that minimize the energy consumption while guaranteeing that comfort requirements are met. A comprehensive and up-to-date overview of the literature related to the predictive control of buildings can be found on the website of the OptiControl project (`www.opticontrol.ethz.ch`). From the wide variety of results, a few instances can be listed. The controller *i)* takes disturbance predictions (occupancy, weather etc.) into account, thus it adjusts control actions appropriately [46, 30], *ii)* can utilize the thermal mass of a building in a better way compared to the conventional control strategies (e.g PID, weather compensated or rule based control) [4, 3], *iii)* is able to deal with variable energy price that can be easily included into the formulation of the optimization problem [2, 23], *iv)* can handle minimization of the energy peaks and thus shift energy loads within certain time frame [21, 41, 14, 31] (beneficial because of both the possibility of tariff selection and lowering operational costs), *v)* can take into account stochastic properties of random disturbance variables (e.g. weather forecast, occupancy profiles); convex approximation of a stochastic model predictive control problem for buildings is given in [32], *vi)* can be formulated in a distributed manner and thus the computational load can be split among several solvers [44, 27, 17]. There have also been reported some experimental setups of MPC which have shown the energy

savings potential [21, 46, 24, 35](15–30 % compared to conventional control strategies).
The increased popularity of MPC usage for building control in recent years is indisputable, however, most of the results are based on the simulations or short time experiments. In this manuscript, we provide a detailed description of an MPC implementation on a real building and we analyze results from two months of operation.
The text is further organized as follows. The predictive control strategy is presented in Section 4.2. Section 4.3 is devoted to modeling. Case-study is discussed in Section 4.4. Finally, the text is concluded by Section 4.5 .

## 4.2 Model predictive control

The Building Automation System (BAS) aims at controlling heating, cooling, ventilation, blind positioning, and electric lighting, of a building such that the temperature, $CO_2$ and luminance levels in rooms or building zones stay within the desired comfort ranges. One typically divides the control hierarchy into two levels: the low-level controller which typically operates at the room-level and is used to track a specified setpoint, and a high-level controller which is done for the whole building and determines the setpoints for the low-level controllers. The article focuses on the usage of Model Predictive Control (MPC), which is used as high-level controller.

### 4.2.1 MPC strategy

MPC is a method for constrained control which originated in the late seventies and early eighties in the process industries (oil refineries, chemical plants, etc.) (see e.g. [39, 34, 40, 16]). MPC is not a single strategy, but a class of control methods with the model of the process explicitly expressed in order to obtain a control signal by minimizing an objective function subject to some constraints. In building control one would aim at optimizing the energy use or cost subject to comfort constraints.
MPC is a very simple and satisfyingly intuitive approach to constrained control. During each sampling interval, a finite horizon optimal control problem is formulated and solved over a finite future window. The result is a trajectory of inputs and states into the future satisfying the dynamics and constraints of the building while optimizing some given criteria. In terms of building control, this means that at the current point in time, a heating/cooling etc. plan is formulated for the next several hours to days, based on predictions of the upcoming weather conditions. Predictions of any other disturbances

(e.g., internal gains), time-dependencies of the control costs (e.g., dynamic electricity prices), or of the constraints (e.g., thermal comfort range) can be readily included in the optimization.
The first step of the control plan is applied to the building, setting all the heating, cooling and ventilation elements, then the process moves one step forward and the procedure is repeated at the next time instant. This receding horizon approach is what introduces feedback into the system, since the new optimal control problem solved at the beginning of the next time interval will be a function of the new state at that point in time and hence of any disturbances that have acted on the building.

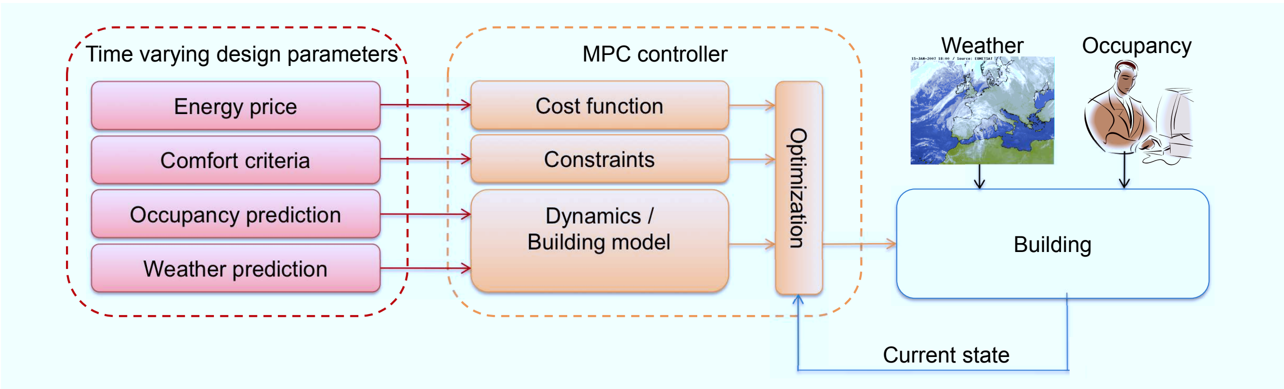


Figure 4.1: Basic principle of Model Predictive Control for Buildings.

Fig. 4.1 summarizes the basic MPC control scheme. As time-varying design parameters, the energy price, the comfort criteria, as well as predictions of the weather and occupancy are input to the MPC controller. One can see that the modeling and design effort consist of specifying a dynamic model of the building, as well as constraints of the control problem and a cost function that encapsulates the desired behavior. In each sampling interval, these components are combined and converted into an optimization problem depending on the MPC framework chosen. A generic framework is given by the following finite-horizon optimization problem:

**Problem 1:**

$$\min_{u_0,\dots,u_{N-1}} \sum_{k=0}^{N-1} l_k(x_k, u_k) \qquad \text{Cost function} \tag{4.1}$$

subject to

$$x_0 = x \qquad \text{Current state} \tag{4.2}$$

$$x_{k+1} = f(x_k, u_k) \qquad \text{Dynamics} \tag{4.3}$$

$$(x_k, u_k) \in \mathcal{X}_k \times \mathcal{U}_k \qquad \text{Constraints} \tag{4.4}$$

where $x_k \in \mathbb{R}^n$ is the state, $u_k \in \mathbb{R}^m$ is the control input, $k$ is the time step, $\mathcal{X}_k$ and $\mathcal{U}_k$ denote the constraints sets of the state and inputs respectively and are explained below. We now detail each of the four components in the above MPC formulation and discuss how they affect the system and the resulting optimization problem. Please note that this is not a comprehensive overview of MPC formulations, but rather a collection of formulations, which are frequently used or reasonable in the field of building control. For a more comprehensive overview on MPC formulations, the reader is referred e.g. to [25].

### Cost function

The cost function generally describes stability and performance targets. The cost is generally, but not always, used to specify a preference for one behavior over another, e.g., minimum energy or maximum comfort.
Generally, the main goal is to minimize energy cost while respecting comfort constraints, which can be formalized by the following cost function:

$$l_k(x_k, u_k) = (y_k - y_{r,k})^T Q_k (y_k - y_{r,k}) + R_k u_k, \tag{4.5}$$

where $Q_k$ and $R_k$ are time varying matrices of appropriate size and $y_{r,k}$ the reference signal at time $k$. The trade-off between precision of reference tracking and energy consumption is expressed by proportion of the matrices $Q_k$ and $R_k$. The reference tracking is expressed as a quadratic form because it significantly penalizes larger deviations from the reference. The energy bill is usually an affine function of a total amount of consumed energy. Therefore, the control cost is weighted linearly.

### Current state

The system model is initialized to the measured/estimated current state of the building and all future (control) predictions begin from this initial state $x$. Depending on what the state of the building is describing, it might not be possible to measure everything directly. In this case, a Kalman filter can be used to estimate the current state of the building and the estimate is used as initial state.

### Dynamics

The controller model, i.e. the mathematical description of the building dynamics is a critical piece of the MPC controller. For the work presented in this paper we restrict ourselves to linear dynamics. This is the most common model type and the only one that will result in a convex and easily solvable optimization problem.

### Constraints

The ability to specify constraints in the MPC formulation and to have the optimization routine handle them directly is the key strength of the MPC approach. There can be constraints on the states or the output, as well as on the input. When explaining different forms of constraints in the following we will do it for input constraints only, but everything applies for state and output constraints alike. *Linear constraints* are the most common type of constraint, which are used to place upper/lower bounds on system variables

$$u_{min,k} \leq u_k \leq u_{max,k}, \tag{4.6}$$

or generally formulated as

$$G_k u_k \leq g_k. \tag{4.7}$$

The constraints can be constant, given by physical or logical limitations. For instance, valve cannot be open more that 100% or temperature of heating water cannot exceed some predefined level. The constraints can be also time-varying, e.g. to account for different comfort constraints during day-time and night-time. In general case, the constraints can be a function of state variables or inputs.. This class of constraints can also be used to approximate any convex constraint to an arbitrary degree of accuracy. Linear constraints also result in the simplest optimization problems. Furthermore, one might want to constrain the rate of change, which is done by imposing a constraint of the form

$$|u_k - u_{k-1}| \leq \Delta_{u_{max}}. \tag{4.8}$$

## 4.2.2 Optimal control formulations for buildings

In the following discussion we restrict ourselves on the deterministic centralized MPC formulations because such formulations are the most widely used one in practice and moreover stochastic or distributed MPC would only bring some additional complexity.

We also assume that models of the buildings are linear time invariant (LTI) and that they have heat fluxes as the system inputs while the zone temperatures are system outputs. The models have following form:

$$x_{k+1} = Ax_k + Bu_k + Vv_k, \tag{4.9}$$

$$y_k = Cx_k + Du_k + Wv_k. \tag{4.10}$$

$v_k \in \mathbb{R}^s$ is a vector of disturbances, $y_k \in \mathbb{R}^p$ is a vector of system outputs. Matrices $A, B, C, D, V, W$ are so called system matrices and are of appropriate dimensions.

We will start from the formulations that have appeared in the literature. Pros and cons for each of the formulation will be given. Each formulation eliminates some drawbacks of the previous one and the last one is *hopefully* the most suitable formulation for buildings (both from the point of view of the quantities being optimized and practical viewpoint).

### Minimization of delivered energy and satisfaction of the constraints

This formulation was reported by [10]. The cost function contains only a term standing for the minimization of the delivered energy while the thermal comfort is guaranteed by means of hard constraints on the system outputs, i.e. zone temperatures.

$$\begin{aligned}
&\min_{u} \sum_{k=0}^{N_u-1} |R_k u_k|_1 \\
&\text{subject to:} \\
&F_k x_k + G_k u_k \leq h \\
&x_{k+1} = Ax_k + Bu_k + Vv_k \quad k = 1 \ldots N_y \\
&y_k = Cx_k + Du_k + Wv_k \quad k = 1 \ldots N_y \\
&x_0 = x_{init} \\
&\underline{r}_k \leq y_k \leq \overline{r}_k \quad k = 1 \ldots N_y
\end{aligned}$$

$F_k, G_k, h_k$ define time varying polytopic constraints on system inputs and states, while $\underline{r}_k$ and $\overline{r}_k$ stand for the time varying reference trajectory for the system outputs. Initial state $x_{init}$ is a parameter of the optimization and is provided by means of Kalman filter or full state measurement at each control timestep.

Although the presented control strategy was presented as a *new control strategy suitable for MPC for buildings*, such a optimal control problem formulation cannot be used in the practice. At least from the one reason: if the initial state implies any comfort violation then such an optimization problem will be infeasible and the controller cannot work anymore. Feasibility issues are usually handled with the aid of so called slack variables on system states and system outputs. Hard constraints are imposed only on the system inputs, i.e. decision variables. More details on this topic are given in the following subsection.

### Trade-off between energy consumption and comfort violations

Slack variables are additional decision variables that are being weighted only in situations when some quantity, which the slack variable is imposed on, reaches certain bound. They are usefull especially in situations when the objective is to keep system outputs within a certain range – only violation of the range is penalized.

$$
\begin{aligned}
\min_{u} \sum_{k=0}^{N_u-1} |R_k u_k|_1 &+ \sum_{k=0}^{N_y-1} |Q_k(y_k - z_k)|_2^2 \\
\text{subject to:}& \\
F_k x_k + G_k u_k &\leq h \\
x_{k+1} &= Ax_k + Bu_k + Vv_k \quad k = 1 \dots N_y \\
y_k &= Cx_k + Du_k + Wv_k \quad k = 1 \dots N_y \\
x_0 &= x_{init} \\
\underline{r}_k &\leq z_k \leq \overline{r}_k \quad k = 1 \dots N_y
\end{aligned}
$$

In this optimal control problem setup $z_k \in \mathbb{R}^p$ is the slack variable on the zone temperature.

The advantages of such a formulation has already been discussed, however, one norm weighting the system inputs belong among the biggest disadvantages. As it is well known, the solution of a linear program lies in some of the vertex of the polytopic constraints. If the constraints are not very tight the control results into either idle (no energy is delivered) or deadbeat control (maximum value of the energy is supplied). This behavior causes issues especially in closed loop performance. If there is some model mismatch, then the control actions might result in a very oscillatory behavior. Unpleasant oscilations can be suppressed by introducing hard constraints on the maximum rate of change of the input signals. But what if, accidentally, there is a strong need to heat up the building and to use the maximum capacity of the heating system immediately? Therefore soft constraints on the rate of change are chosen. Details are given in the following subsection.

### Practical aspects motivated formulation

The formulation with the soft constraints on the maximum rate of change of the input signal is as follows:

$$\min_{u} \sum_{k=0}^{N_u-1} \left( |R_k u_k|_1 + \delta \, |u_k - u_{k-1} - p_k|_2^2 \right) + \sum_{k=0}^{N_y-1} |Q_k(y_k - z_k)|_2^2$$

subject to:

$$\begin{aligned}
F_k x_k + G_k u_k &\leq h_k \\
x_{k+1} &= A x_k + B u_k + V v_k \quad k = 1 \dots N_y \\
y_k &= C x_k + D u_k + W v_k \quad k = 1 \dots N_y \\
u_{-1} &= u_{last} \\
x_0 &= x_{init} \\
\underline{r}_k \leq z_k &\leq \overline{r}_k \quad k = 1 \dots N_y \\
\underline{\Delta u} \leq p_k &\leq \overline{\Delta u} \quad k = 1 \dots N_u
\end{aligned}$$

$\underline{\Delta u}, \overline{\Delta u}$ are minimum/maximum values allowed for the input change not to be penalized, whilst $u_{last}$ is the system input from the previous control timestep.

## 4.3 Modeling

Modeling of the building requires insight both into control engineering as well as into HVAC engineering. Moreover, it is also the most time demanding part of designing the MPC setup.

Three approaches to building modeling are outlined in this section. Two of them come from so-called RC modeling, the other one is purely black box technique called Subspace identification method. The aim is to provide insight into these techniques with emphasis on their applicability for MPC. Largely used computer aided modeling tools (e.g. TRNSYS, EnergyPlus, ESP-r etc.) are not considered here, as they result in complex models which cannot be readily used for control purposes.

When large measurement data sets are available, a purely statistical approach for creation of a building model is preferred. A large number of System Identification methods exists (a survey is listed in e.g. [20]), however, only a few of them have the capability of identification of multiple-input multiple-output (MIMO) systems, which are considered in case of building control. For identification of linear MIMO models, subspace identification methods are often used [20, 49, 48] and have been suggested for identification of building

models as in [36].
Alternatively to the statistical approach, especially if there is a lack of data or some knowledge of building physics is present, the RC modeling can be used.

### 4.3.1 RC modeling

The principle of the thermal dynamics modeling can easily be described by a small example as given in Fig. 4.2. The room can be thought of as a network of first-order systems, where the nodes are the system states and these represent the room temperature or the temperatures in the walls, floor or ceiling. Then the heat transfer rate is given by

$$\begin{aligned} \frac{dQ}{dt} &= K_{ie} \cdot (\vartheta_e - \vartheta_i) \\ \Rightarrow \underbrace{\frac{dQ}{d\vartheta_i}}_{C_i} \cdot \frac{d\vartheta_i}{dt} &= K_{ie} \cdot (\vartheta_e - \vartheta_i), \end{aligned} \tag{4.11}$$

where $t$ denotes the time, $\vartheta_i$ and $\vartheta_e$ are the temperatures in nodes $i$ and $e$ respectively, $Q$ is thermal energy, and $C_i$ denotes the thermal capacitance of node $i$. The total heat transmission coefficient $K_{ie}$ is computed as

$$\frac{1}{K_{ie}} = \frac{1}{K_i} + \frac{1}{K_e}, \tag{4.12}$$

where the heat transmission coefficients $K_i$ and $K_e$ depend on the materials of $i$ and $e$ as well as on the cross sectional area of the heat transmission. For each node, i.e. state, such a differential equation as in Eq. (4.11) is formulated. The actuators are direct inputs to the node, which means that their input is added. The modeling of illumination and $CO_2$ concentration is omitted here for brevity, for more details on RC modeling see [9].
The model parameters (e.g. $K_{ie}$ or $C_i$ in Eq. (4.11)) can be determined in two ways: by reading from construction plans or by statistical estimation, which is described in the next sections.

#### Construction plan

Thermal capacities, resistances and other unknown parameters are determined from the construction plan according to the materials used and their tabular values. Simulations of the acquired model are then required to validate the model accuracy. If the model does not correspond to the measured data, parameter adjustment is necessary.

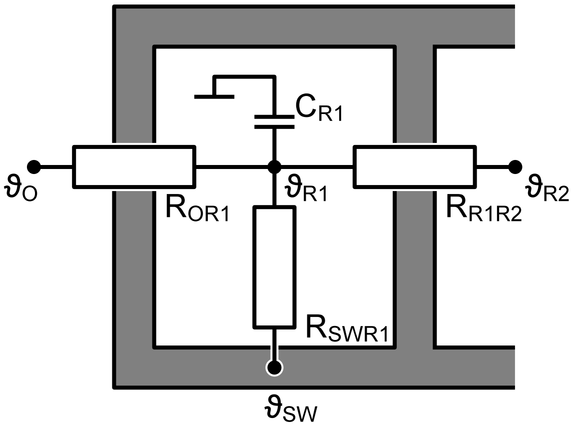


Figure 4.2: RC modeling is based on the description of heat transmission between nodes that are representing temperatures. The figure captures example with two rooms where, $\vartheta_{R1}$ and $\vartheta_{R2}$ are the temperatures in the room $R1$ and $R2$, respectively, $\vartheta_0$ is the outside temperature, $\vartheta_{SW}$ is the temperature of the supply water used for floor heating, $C_{R1}$ denotes the thermal capacity of the room $R1$. Resistances are representing the thermal resistances between the nodes.

## Statistical estimation

Having described the physics of the building by a set of differential equations, the estimation problem is formulated in the continuous time. Most of the mathematical tools, however, work with the discrete-time counterparts, therefore the original continuous-time problem must be reformulated to the discrete world, e.g. as

$$A = e^{A_c T_s} = I + A_c T_s + \frac{A_c^2 T_s^2}{2} + \ldots \approx I + A_c T_s,$$
$$B = \int_0^{T_s} e^{A_c \tau} d\tau \approx \int_0^{T_s} I d\tau B_c = T_s B_c,$$

where $A_c, B_c$ and $A, B$ are model matrices of continuous- and discrete-time models, respectively. $T_s$ stands for sampling time. This corresponds to the Euler's discretization, thus can be applied for non-linear systems as well. Then the state equation can be written as

$$\begin{aligned} X_1^N &= A X_0^{N-1} + B U_0^{N-1} + E_0^{N-1} = \\ &= \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} X_0^{N-1} \\ U_0^{N-1} \end{bmatrix} + E_0^{N-1} \end{aligned} \tag{4.13}$$

with $N+1$ being the number of samples and

$$\begin{aligned} X_0^{N-1} &= \begin{bmatrix} x(0), & x(1), & \ldots, & x(N-1) \end{bmatrix}, \\ U_0^{N-1} &= \begin{bmatrix} u(0), & u(1), & \ldots, & u(N-1) \end{bmatrix}, \\ E_0^{N-1} &= \begin{bmatrix} e(0), & e(1), & \ldots, & e(N-1) \end{bmatrix}. \end{aligned}$$

For standard optimization using OLS, Eq. (4.13) is rewritten as

$$\mathrm{vec}X_1^N = \left( \begin{bmatrix} X_0^{N-1} \\ U_0^{N-1} \end{bmatrix} \otimes I_n \right)^T \mathrm{vec} \begin{bmatrix} A & B \end{bmatrix} + \mathrm{vec}E_0^{N-1} \tag{4.14}$$

with $I_n$ being $n \times n$ identity matrix, $n$ represents system order, (vec $\bullet$) is vectorization of a matrix and ($\bullet \otimes \bullet$) is a Kronecker product. Extra lines for the structure preservation of $A$ and $B$ as well as other required constraints can be added into the regressor matrix and left-hand side matrix. Then, the unknown parameters are estimated using weighted least squares technique.

### 4.3.2 Subspace identification algorithm

One of the most powerful contributors to the quality of the predictive control is a well identified model. There are several completely different approaches to the system identification including physical modeling (e.g. computational fluid dynamics (CFD) modeling [28]) or statistical identification. As traditional methods are, for buildings, rather time consuming, and do not posses the capability of proper handling of MIMO systems, we have turned towards statistical identification methods, and more specifically, towards subspace methods [20, 49, 13].
The objective of the subspace algorithm is to find a linear, time invariant, discrete time model in an innovation form

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + Ke_k \\ y_k &= Cx_k + Du_k + e_k, \end{aligned} \tag{4.15}$$

where $A$, $B$, $C$, and $D$ are system matrices, $K$ is Kalman gain – derived from the Algebraic Riccati Equation (ARE) ([15]), and $e$ is a white noise sequence. This model is equivalent to the well-known stochastic model as defined in e.g. [18, 12]. The objective of the algorithm is to determine the system order $n$ and to find the system as well as state and measurement noise covariance matrices given the sequence of input $u(k)$ and output $y(k)$ measurements.
The main difference between classical and subspace identification is, given the input and output data, as follows:

- **Classical approach.** Find the system matrices, then estimate the system states, which often leads to high order models that have to be reduced thereafter.
- **Subspace approach.** Use orthogonal and oblique projections to find Kalman state sequence (see [15]), then obtain the system matrices using least squares method.

The entry point to the algorithm are input-output equations as follows:

$$\begin{aligned} Y_p &= \Gamma_i X_p^d + H_i^d U_p + Y_p^s \\ Y_f &= \Gamma_i X_f^d + H_i^d U_f + Y_f^s \\ X_f^d &= A^i X_p^d + \Delta_i^d U_p, \end{aligned} \tag{4.16}$$

where $Y_p$ and $Y_f$ are the Hankel matrices of past and future outputs, $U_p$ and $U_f$ are the Hankel matrices of past and future inputs, $X_p^d$ and $X_f^d$ are the deterministic Kalman state sequences, $Y_p^s$ and $Y_f^s$ are the stochastic Hankel matrices of past and future outputs, $H_i^d$ is the lower block triangular Toeplitz matrix for the deterministic subsystem (which contains system matrices), $\Gamma_i$ is the extended system observability matrix (which contains matrices $A$ and $C$) and $\Delta_i^d$ is the deterministic reversed extended controllability matrix (which contains matrices $A$ and $B$). Detailed construction of state matrices can be found in [20, 49]. It is quite straightforward that following holds:

$$\begin{bmatrix} \hat{X}_{i+1} \\ Y_{i|i} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \hat{X}_i \\ U_{i|i} \end{bmatrix} + \begin{bmatrix} \rho_w \\ \rho_v \end{bmatrix} \tag{4.17}$$

$\hat{X}_i$ denotes estimate of state sequence, $\rho_w$ and $\rho_v$ are Kalman filter residuals. State sequence estimates are determined as follows:

$$\begin{aligned} \hat{X}_i &= \Gamma_i^\dagger \left[\mathfrak{Z}_i - H_i^d U_f\right] \\ \hat{X}_{i+1} &= \Gamma_{i-1}^\dagger \left[\mathfrak{Z}_{i+1} - H_{i+1}^d U_f^-\right], \end{aligned} \tag{4.18}$$

with $\mathfrak{Z}_i$ and $\mathfrak{Z}_{i+1}$ defined as oblique projections ([49])

$$\begin{aligned} \mathfrak{Z}_i &= Y_f \underset{U_f}{/} W_p \\ \mathfrak{Z}_{i+1} &= Y_f^- \underset{U_f^-}{/} W_p^+. \end{aligned} \tag{4.19}$$

Solving Eq. (4.17) using least squares methods, we get the state space system description of the system, namely the system in the innovation form (Eq. (4.15)).

The essential condition for optimal filter run is knowledge of the noise covariance matrices $Q$ and $R$ (state and measurement noise covariance matrices[1]). These two matrices are used for calculation of Kalman gain $K$. In early 70s' Mehra's publications on covariance matrices estimation were published [26]. Then, for a large period, the estimation of covariance matrices was largely overviewed and only in 2006 Odelson's article [29] was published that offered

[1] $S$ can be considered zero, because any non-zero $S$ can be easily transformed to zero matrix [19]

Table 4.1: Comparison of the identification/modeling approaches.

| | Building simulation software modeling | RC modeling – tabular data driven | RC modeling – statistical estimation | Statistical Identification |
|---|---|---|---|---|
| Planing data from architects and engineers need | yes | yes | no | no |
| Operation data need | no | no | yes | yes |
| HVAC engineering background needed | yes | yes | no | no |
| Result is achieved in defined time | yes | yes | no | no |
| Use of prior information about building | yes | yes | yes | no |
| Continuous model update | no | no | yes | no |
| MPC applicable | no | yes | yes | yes |

Figure 4.3: The building of the Czech Technical University in Prague that was used for MPC application

a new method for $Q$ and $R$ estimation called Autocovariance Least-Squares (ALS) technique ([29]). A few more modifications of this method can be found in [38, 1]. Kalman gain matrix $K$ is computed in a standard way using state and noise covariance matrices computed using ALS as described in [29].

### 4.3.3 Comparison of the identification approaches

Finally, Table 4.1 summarizes the MPC applicability of above mentioned approaches.

## 4.4 Case study

The presented MPC scheme of Problem 1 was applied to the building heating system of the Czech Technical University (CTU) in Prague, see Fig. 4.3. MPC was applied there from January 2010 and was operational until the end of heating season in mid-March 2010.

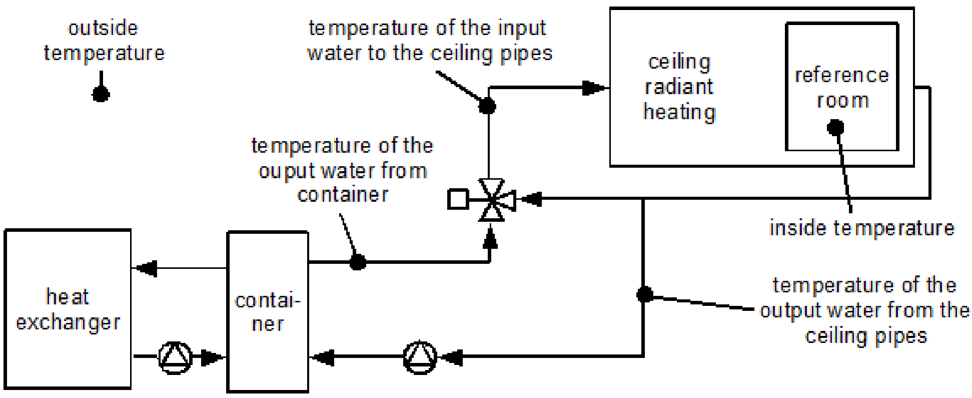


Figure 4.4: Simplified scheme of the ceiling radiant heating system

### 4.4.1 Description of the building

The building of the CTU uses Crittall [5] type ceiling radiant heating and cooling system. In this system, the heating (or cooling) beams are embedded into the concrete ceiling. A simplified scheme of the ceiling radiant heating system is illustrated in Fig. 4.4. The source of heat is a vapor-liquid heat exchanger, which supplies the heating water to the water container. A mixing occurs here, and the water is supplied to the respective heating circuits. An accurate temperature control of the heating water for respective circuits is achieved by a three-port valve with a servo drive. The heating water is then supplied to the respective ceiling beams. There is one measurement point in a reference room for every circuit. The set-point of the control valve is therefore the control variable for the ceiling radiant heating system in each circuit.

#### Modeling of the building block

The ceiling radiant heating system was modeled by a discrete-time linear time invariant stochastic model. We can consider this model as a Kalman filter giving an estimates of the state and the output denoted as $\hat{x}_k$ and $\hat{y}_k$. Outside temperature prediction[2] and heating water temperatures were used as the model inputs. The prediction of outside temperature is composed of two values, $T_{max}$ and $T_{min}$, defining a confidence interval. The outputs of the model are estimates of the inside temperature $\hat{T}_{in}$ and the temperature

[2] Acquired from National Oceanic and Atmospheric Administration (NOAA), http://www.noaa.gov

of the return water[3] $\hat{T}_{rw}$. This can be formalized as

$$\begin{aligned} \hat{x}_{k+1} &= A\hat{x}_k + B \begin{bmatrix} T_{min,k} \\ T_{max,k} \\ T_{hw,k} \end{bmatrix} + Ke_k \\ \begin{bmatrix} \hat{T}_{in,k} \\ \hat{T}_{rw,k} \end{bmatrix} &= C\hat{x}_k + D \begin{bmatrix} T_{min,k} \\ T_{max,k} \\ T_{hw,k} \end{bmatrix}, \end{aligned} \tag{4.20}$$

where $T_{hw}$ is a temperature of the heating water and $T_{in}$ denotes the inside temperature, $e_k = y_k - C\hat{x}_k - Du_k$ with $y_k$ denoting real output (comprises temperature of the room and temperature of the return water) and $x_k$ denoting system state. System matrices $A$, $B$, $C$ and $D$ are to be identified using subspace methods. The state $\hat{x}_k$ has no physical interpretation, when identified by means of the subspace identification. System order is determined by the identification algorithm as well. Modeling of the heating system of the CTU building is discussed in detail in [7].

## 4.4.2 Control objectives

There are several requirements to be fulfilled:

### Reference tracking

The reference trajectory $y_{r,k}$, room temperature in our case, is known prior, as a schedule. The major advantage of MPC is the ability of computing the outputs and corresponding input signals in advance, that is, it is possible to avoid sudden changes in control signal and undesired effects of delays in system response.

Our aim of the control is that the room temperature should adhere the upper desired value from its beginning to its end, whilst the lower reference level is not important until the room temperature approaches significantly to it. Then the lower level should be tracked too. This behavior will be achieved by means of slack variables on the system outputs, i.e. zone temperatures.

### Minimization of energy consumption

As the return water circulates in the heating system (see Fig. 4.4), energy consumed by the heating-up of the building is linearly depended on the pos-

[3] It is crucial to model return water as an output because it gives a significant information about energy accumulated in the building, moreover it represents the interconnection between heating water and room temperature. Omitting the return water would lead to significant lost of information.

itive difference between heating $T_{hw}$ and return water $T_{rw}$ temperatures entering/exiting the three port valve in Fig. 4.4. Thus, the 1-norm of weighted inputs is to be minimized.
It is worthy to note, that the deterministic formulation of MPC was used, although the model given by subspace identification algorithm was stochastic. Since the noise $e_k$ in Eq. (4.15) is supposed to be zero mean, one can take only deterministic part of the model defined by Eq. (4.15) for computation of optimal system input.

### 4.4.3 MPC problem formulation

At first, the deterministic part of the given system from Section 4.3.2 is partitioned as follows:

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ y_{1,k} &= C_1x_k + D_1u_k \\ y_{2,k} &= C_2x_k + D_2u_k, \end{aligned}$$

where $y_{1,k}$ stands for outputs with reference signal (e.g. $T_{in,k}$), whilst $y_{2,k}$ represents the input-output differences – in our case $y_{2,k} = T_{hw,k} - T_{rw,k}$.
The requirements (see Section 4.4.2) for the weighting of the particular variables can be carried out by adding slack variables $a(k) \in \dim y_{1,k}$ and $b_k \in \dim y_{2,k}$. The resulting optimization problem can be written as follows:

$$\begin{aligned} J = \min_{a_k,b_k,u_k} \sum_{k=0}^{N-1} \|Q_1a_k\|_2^2 + \|Q_2b_k\|_1 \\ y_{r,k} - y_{1,k} - a_k &\leq 0, \quad a \geq 0 \\ y_{2,k} - b_k &\leq 0, \quad b \geq 0 \\ u_{min} \leq u_k &\leq u_{max} \\ |u_k - u_{k-1}| &\leq \Delta_{u_{max}} \\ y_{1,k} = C_1A^{k-1}x_0 + \sum_{i=0}^{k-1} C_1A^{k-i-1}Bu_i + D_1u_k \\ y_{2,k} = C_2A^{k-1}x_0 + \sum_{i=0}^{k-1} C_2A^{k-i-1}Bu_i + D_2u_k. \end{aligned} \tag{4.21}$$

$Q_1$ and $Q_2$ stand for weighting matrices of appropriate dimension, $u_{min}$ and $u_{max}$ represent lower and upper bounds of the input signals and $\Delta_{u_{max}}$ is maximum rate of change of the input signal.
Eq. (4.21) can be readily rewritten into quadratic programming (QP) problem and solved using any QP solver.

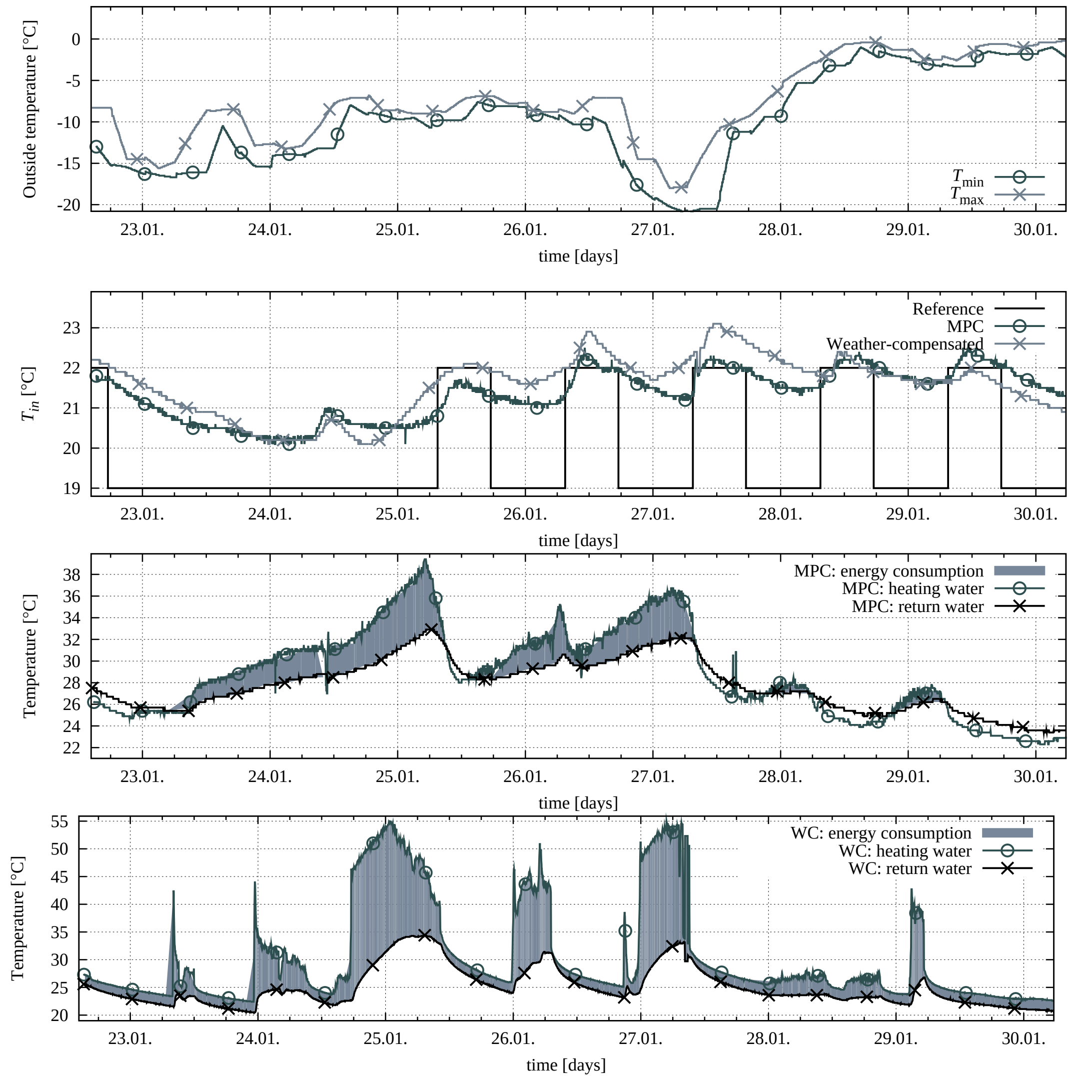

Figure 4.5: Different control strategies: comparison of weather-compensated (WC) and predictive control (MPC) of heating water temperature and the room temperature controlled by MPC.

## 4.4.4 Results

Two nearly identical blocks of the CTU building were used for testing. The first block was controlled by weather-compensated controller, while the second one by the predictive controller. Three month of real operation were used for investigation of controllers' performance and savings, nine days segment of the period is depicted in Fig. 4.5. The upper part shows outside temperature, whilst the lower compares reference tracking for weather-compensated and predictive controllers. It can be seen, that the predictive controller heats in advance in order to perform optimal reference tracking, that is, inside comfort, and minimum energy consumption. Two last subfigures compare the efficiency of control measured by energy consumption.

Table 4.2: Comparison of heating curve (HC) and model predictive control (MPC) strategies using similar building blocks $B_1$ and $B_2$.

| | | $B_1$ | | $B_2$ | | |
|---|---|---|---|---|---|---|
| | mean $\vartheta_o$ [°C] | control | mean $\vartheta_s, \vartheta_n$ [°C] | control | mean $\vartheta_s, \vartheta_n$ [°C] | MPC savings |
| $1^{st}$ week | -3.4 | HC | 21.4 | MPC | 21.1 | 15.54 % |
| $2^{nd}$ week | -1.3 | MPC | 21.4 | HC | 20.9 | 16.94 % |

### Evaluation of MPC energy savings

Evaluation of the energy savings achieved by different control strategies is a complicated task. The weather conditions change all the time, as well as the number and behavior of the building occupants. Single comparisons of results are affected by these disturbances, therefore one independent comparison of the real building experiment will be presented.
The comparison denoted as cross comparison uses almost similar building blocks $B_1$ and $B_2$. The cross comparison had two phases, each lasted for a week. In the first week, block $B_1$ was controlled by the heating curve and block $B_2$ by MPC. The other week, the control strategies were switched. The advantage of the cross comparison is compensation of the majority of disturbances because both building blocks are exposed to the same weather conditions.
The cross comparison results are summarized in Table 4.2. According to this comparison, MPC saved approximately 16% of energy in both weeks.
The efficiency of the predictive control was superior to the weather-compensated controller, even if the active heating was necessary. In fact, the costs were about 16% lower in case of predictive controller. The main reasons of the savings is the ability of predictive controller to fully employ the thermal capacitance of a building and that the character of an optimal input is not so aggressive in comparison to conventional control strategy. Predictive controller circumspect strategy of the input signal shaping reduces the maximum peak of the heating water as well. Consequently, the less expensive tariff of a primary energy source could also contribute to further savings.

## 4.5 Conclusions

Predictive control has a great potential in the area of building control especially in case of buildings with great heat accumulation capabilities. Testing confirmed our empirical experiences and the efficiency of the predictive controller in comparison to weather-compensated controller.
MPC implementation, and foremost the modeling effort presents, presents

the most time consuming part of MPC integration into a building automation systems. In contrast to the current building control techniques, MPC is based on a non trivial mathematical background that complicates its usage in practice. But its contribution to reduction of a building operation cost is so significant that it is expected that it will become a common solution for so-called intelligent buildings in a few years.

## 4.6 Acknowledgements

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- Cigler J., Prívara S. Subspace Identification and Model Predictive Control for Buildings. In: *Proceedings of 11th International Conference on Control, Automation, Robotics and Vision*, 2010.
- Široký J, Oldewurtel F, Cigler J, Prívara S. ?Experimental analysis of model predictive control for an energy efficient building heating system. *Applied Energy*. 2011;88(9):1-9

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# Chapter 5

# Stochastic model predictive control

Ing. Jiří Cigler

*In the previous chapter, the basics of the modeling and optimal control problem formulation were proposed in a deterministic fashion. However, disturbances acting on the system may have a stochastic nature. For instance, weather forecast has a predefined level of uncertainty [14], occupancy profiles are not exactly known in advance but can be considered as a random variables with known distribution [20], etc. Moreover, probability of thermal comfort constraints satisfaction belongs among the requirements defined by ISO 7730 norm (concerning thermal comfort in buildings [12]). These facts motivate researchers to investigate more the stochastic model predictive formulation that can be afterwards applied to building climate control. In the following text, a convex approximation to one norm stochastic model predictive control problem will be presented. This algorithm can be readily used for small-scale building problems.*

## 5.1 Introduction

Stochastic control is a relatively mature field, yet there is still a considerable number of unresolved problems mostly due to the notorious inherent intractability of the vast majority of them. Only a handful of stochastic optimal control problems (e.g. the linear quadratic control) can be solved optimally, whereas the remainder has to be tackled by various approximation techniques most frequently arising from the dynamic programming paradigm [1], [17].
Recent advances in computation and mathematical optimization techniques have, however, opened new ways of dealing with these problems. One of the simplest, yet in most practical applications very effective approach, is the certainty equivalent model predictive control (CE-MPC) [2], [1] that solves a deterministic optimization problem with stochastic disturbances replaced by their estimates based upon the information available at the time, and proceeds in a receding horizon fashion. Another popular class of control strategies is the affine disturbance feedback policy which turns out to be

equivalent to the affine state feedback policy via a nonlinear transformation similar to the classical Q-design or Youla-Kučera parametrization [18], [19]. However convenient the paradigm of affine disturbance feedback may be, its use is prohibitive whenever unbounded stochastic disturbances enter the system in the presence of hard control input bounds since then the linear part necessarily vanishes, which, in effect, renders the policy open loop. One way to overcome this problem is to use a saturated nonlinear disturbance feedback as in [10], where this approach was developed for the quadratic cost. In this article we follow up on this work and develop a methodology for solving this problem in the 1-norm with the additional assumption of the disturbances being jointly Gaussian (but not necessarily independent).
Another branch of approximation techniques bounds the disturbances a priori and solves a robust MPC problem, while guaranteeing an open loop probabilistic bound on the performance [4]. This approach, however, tends to be very conservative, and thus the idea of bounding the disturbances a priori based on their distribution appears more often in the context of chance constraints, see e.g. [15]. For different approaches to chance constraints handling see [5], [13].
The very important, though much neglected, question of stability and recursive feasibility of stochastic receding horizon schemes is addressed in a series of papers [7], [8], [9] and [16]. These papers, however, assume either compactly supported disturbances or only probabilistic input and state constraints, whereas [10] and [11] deal exclusively with stability in the presence of hard input constraints. In this paper we prove in a much simpler way a slight generalization of one of their stability results.

### 5.1.1 Notation

Throughout the article $\mathbb{R}$ denotes the set of reals, $N$ and $N_c$ denote the prediction and control horizons, respectively. The positive integers $m$ and $n$ denote the number of control inputs and the state-space dimension. The function $\mathrm{sat}_r(\cdot)$ denotes the standard elementwise saturation of the components of a vector to $r$, and $||\cdot||_\infty$ denotes the induced infinity norm of a matrix (in particular *not* the maximum absolute value if the matrix is a row vector). $\rho(\cdot)$ and $\mathrm{tr}(\cdot)$ denote the spectral radius and the trace of a square matrix. $\mathbf{E}(\cdot)$ denotes the expectation of a random variable, and $X \sim \mathcal{N}(\mu, \Sigma)$ indicates that $X$ is a Gaussian random variable with the expectation $\mu$ and the covariance matrix $\Sigma$. The symbols $\mathrm{vec}(\cdot)$ and $\otimes$ denote the vectorization and the Kronecker product respectively. Finally, $\mathrm{Hess}(\cdot)$ and $\mathrm{Jac}(\cdot)$ denote the Hessian and the Jacobian of a function.

## 5.2 Problem statement

This article deals with the problem of minimizing the cost function

$$J := \mathbf{E}\left\{||Q_N x_N||_1 + \sum_{k=0}^{N-1} ||Q_k x_k||_1 + ||R_k u_k||_1\right\} \tag{5.1}$$

subject to the discrete-time system dynamics

$$x_{k+1} = Ax_k + Bu_k + w_k, \tag{5.2}$$

$x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$, and hard input constraints

$$||u_k||_\infty \leq U_{\max},\ k = 0, \ldots, N-1, \tag{5.3}$$

where $Q_k \in \mathbb{R}^{n_q \times n}$, $R_k \in \mathbb{R}^{n_r \times m}$ are weighting matrices. All the results derived here generalize with only minor modifications to the case with different bounds on individual control inputs and/or time varying bounds. The disturbances $w = [w_0^T, \ldots, w_{N-1}^T]^T$ are assumed to be jointly Gaussian with the covariance matrix $\Sigma$.

The minimization to be carried out is over all Borel measurable causal disturbance feedback policies

$$u_k = \phi_k(x_0, w_0, \ldots, w_{k-1}),\ k = 0, \ldots, N-1. \tag{5.4}$$

This problem is, however, in general intractable and various approximation techniques exist, see e.g. [1]. For a rigorous treatment of measurability issues in the context of stochastic control see [3]. In this paper, we adopt the approach of [10] where the authors propose to search over a class of causal policies affine in certain nonlinear functions of the disturbances, i.e.

$$u = \eta + Ke(w) = \begin{bmatrix} \eta_0 \\ \vdots \\ \eta_{N-1} \end{bmatrix} + \begin{bmatrix} 0 & 0 & \cdots & 0 \\ K_{1,1} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \\ K_{N-1,1} & \cdots & K_{N-1,N-1} & 0 \end{bmatrix} e(w), \tag{5.5}$$

where $u = [u_0^T, \ldots, u_{N-1}^T]^T$. $\eta \in \mathbb{R}^{mN}$ with blocks in $\mathbb{R}^m$ and strictly lower block triangular $K \in \mathbb{R}^{mN \times nN}$ with blocks in $\mathbb{R}^{m \times n}$ are optimization variables. The choice of the function $e : \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ is discussed later, although it certainly must be bounded should the hard input constraints be satisfied. The bound on $||e(w)||_\infty$ is denoted $\varepsilon$ throughout the article.

One of the main goals of the article is therefore to solve (at least approximately) the optimization problem

$$
\begin{aligned}
\underset{\eta,K}{\text{minimize}} \quad & \mathbf{E}\left\{||Q_N x_N||_1 + \sum_{k=0}^{N-1} ||Q_k x_k||_1 + ||R_k u_k||_1\right\} \\
\text{subject to} \quad & u = \eta + Ke(w) \\
& x_{k+1} = Ax_k + Bu_k + w_k \\
& K \text{ is strictly block lower triangular} \\
& \text{constraints on } \eta, K \text{ such that (5.3) is satisfied.}
\end{aligned}
\tag{5.6}
$$

## 5.3 Main results

The optimization problem (5.6) is, to our knowledge, intractable owing to the 1-norm and the nonlinear function $e(w)$. We therefore propose to solve a relaxed problem where $u = \eta + Ke(w)$ in (5.6) is replaced with $u = \eta + Kw$ while keeping constraints on $\eta$, $K$ such that the hard input constraints are satisfied when the original control policy is used. The relaxed problem must be convex since the objective is convex for each disturbance realization [6]. In the sequel, we show that the relaxed optimization problem is not only convex but also tractable. To this end, we need an expression for the expectation of the absolute value of a Gaussian random variable.

### 5.3.1 Convexity and tractability of the proposed approach

**Lemma 1.** *If $X \sim \mathcal{N}(\mu, \sigma^2)$ then*

$$
g(\mu, \sigma) := \mathbf{E}|X| = \frac{1}{\sqrt{2\pi}}\left(2\,\sigma\, e^{-\frac{\mu^2}{2\sigma^2}} + \mu\sqrt{2\pi}\ \mathrm{erf}\left(\frac{\mu}{\sigma\sqrt{2}}\right)\right) \tag{5.7}
$$

*Proof.* Follows by a straightforward integration from the definition of the expectation of a continuously distributed random variable

$$
\mathbf{E}|X| = \frac{1}{\sigma\sqrt{2\pi}}\left(\int_{-\infty}^{0} -xe^{\frac{-(x-\mu)^2}{2\sigma^2}}\,\mathrm{d}x + \int_{0}^{\infty} xe^{\frac{-(x-\mu)^2}{2\sigma^2}}\,\mathrm{d}x\right), \tag{5.8}
$$

and by using the definition of the error function $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,\mathrm{d}t$. □

Next, we show that the continuous extension (to cater for the $\sigma = 0$ case) of the Gaussian variable modulus expectation is convex under a certain composition and also provide an expression for its gradient and Hessian.

**Lemma 2.** *If $X \sim \mathcal{N}(\mu, \sigma^2)$ for $\sigma > 0$, $X = \mu$ for $\sigma = 0$, and $\mu(\eta, k) = \mu_0 + b^T\eta$, $\sigma(\eta, k) = ||a + Ck||_2$ then the function $f(\eta, k) = (\mathbf{E}|X|)(\eta, k)$ is jointly convex in $(\eta, k)$.*

*Proof.* The proof proceeds directly by computing the Hessian of $f$ for $\sigma > 0$ and then a continuity argument is used to complete the proof. For $\sigma > 0$, $f(\eta, k)$ coincides with $g(\mu(\eta, k), \sigma(\eta, k))$ and the gradient is

$$\nabla f(\mu, \sigma) = \frac{\partial f}{\partial \mu}\nabla\mu + \frac{\partial f}{\partial \sigma}\nabla\sigma = \operatorname{erf}\left(\frac{\mu}{\sigma\sqrt{2}}\right)\nabla\mu + \sqrt{\frac{2}{\pi}}e^{-\frac{\mu^2}{2\sigma^2}}\nabla\sigma \tag{5.9}$$

with

$$\nabla\mu = \begin{bmatrix} b \\ 0 \end{bmatrix}, \qquad \nabla\sigma = \begin{bmatrix} 0 \\ C^T\frac{a+Ck}{\sigma} \end{bmatrix}. \tag{5.10}$$

The expression for $\nabla\sigma$ follows from the fact that $\nabla||x||_2 = \frac{x}{||x||_2}$ and the multivariate form of the chain rule. Now since $\text{Hess}(f) = \text{Jac}(\nabla f)$ and $\text{Jac}(h\tilde{g}) = \tilde{g}(\nabla h)^T + h\text{Jac}(\tilde{g})$ for real-valued function $h$ and multivariate $\tilde{g}$, it follows that

$$\begin{aligned}\text{Hess}(f) = &\begin{bmatrix} b \\ 0 \end{bmatrix}\left\{\nabla\operatorname{erf}\left(\frac{\mu}{\sigma\sqrt{2}}\right)\right\}^T \\ &+ \begin{bmatrix} 0 \\ C^T\frac{a+Ck}{\sigma} \end{bmatrix}\left\{\nabla\left(\sqrt{\frac{2}{\pi}}e^{-\frac{\mu^2}{2\sigma^2}}\right)\right\}^T + \sqrt{\frac{2}{\pi}}e^{-\frac{\mu^2}{2\sigma^2}}\text{Jac}(\nabla\sigma)\end{aligned} \tag{5.11}$$

with

$$\text{Jac}(\nabla\sigma) = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{||x||_2}C^T\left(I - \frac{xx^T}{||x||_2^2}\right)C \end{bmatrix} \succeq 0, \tag{5.12}$$

where $x = a + Ck$ since, again by the chain rule,

$$\text{Jac}_k\,\nabla\sigma = C^T\,\text{Jac}\,\frac{a+Ck}{||a+Ck||_2} = C^T\left[\text{Jac}\left(\frac{y}{||y||_2}\right)\circ(a+Ck)\right]C, \tag{5.13}$$

where $\circ$ denotes the standard function composition. The remaining two terms in (5.11) are

$$\nabla\operatorname{erf}\left(\frac{\mu}{\sigma\sqrt{2}}\right) = \begin{bmatrix} b \\ 0 \end{bmatrix}\frac{1}{\sigma}\sqrt{\frac{2}{\pi}}e^{-\frac{\mu^2}{2\sigma^2}} - \begin{bmatrix} 0 \\ C^T\frac{a+Ck}{\sigma} \end{bmatrix}\sqrt{\frac{2}{\pi}}\frac{\mu}{\sigma^2}e^{-\frac{\mu^2}{2\sigma^2}}, \tag{5.14}$$

$$\nabla\left(\sqrt{\frac{2}{\pi}}e^{-\frac{\mu^2}{2\sigma^2}}\right) = -\begin{bmatrix} b \\ 0 \end{bmatrix}\sqrt{\frac{2}{\pi}}\frac{\mu}{\sigma^2}e^{-\frac{\mu^2}{2\sigma^2}} + \begin{bmatrix} 0 \\ C^T\frac{a+Ck}{\sigma} \end{bmatrix}\sqrt{\frac{2}{\pi}}\frac{\mu^2}{\sigma^3}e^{-\frac{\mu^2}{2\sigma^2}}. \tag{5.15}$$

Rewriting the Hessian with

$$q := \begin{bmatrix} 0 \\ C^T\frac{a+Ck}{\sigma} \end{bmatrix} \tag{5.16}$$

then yields

$$\text{Hess}(f) = \sqrt{\frac{2}{\pi}}e^{-\frac{\mu^2}{2\sigma^2}}\left(\frac{1}{\sigma}\begin{bmatrix} b \\ -q\frac{\mu}{\sigma} \end{bmatrix}\begin{bmatrix} b \\ -q\frac{\mu}{\sigma} \end{bmatrix}^T + \text{Jac}(\nabla\sigma)\right) \succeq 0. \tag{5.17}$$

It is easily seen that $f(\eta, k)$ is continuous and that the sequence of smoothed functions $f_n(\eta, k) = g\left(\mu(\eta, k), \sqrt{\frac{1}{n} + \sum_i x_i^2}\right)$ converges pointwise to $f$. The functions $f_n$ are readily shown to be convex by computing their respective Hessians in the same fashion as above. The function $f(\eta, k)$ is therefore convex since it is a limit of convex functions. $\square$

**Theorem 1.** *The optimization problem*

$$\begin{aligned}
\underset{\eta,K}{\text{minimize}} \quad & \mathbf{E}\left\{||Q_N x_N||_1 + \sum_{k=0}^{N-1} ||Q_k x_k||_1 + ||R_k u_k||_1\right\} \\
\text{subject to} \quad & u = \eta + Kw \\
& x_{k+1} = Ax_k + Bu_k + w_k \\
& K \text{ is strictly block lower triangular} \\
& |\eta_i| + \varepsilon ||K_i||_\infty \leq U_{\max},\ i = 1, \ldots, mN
\end{aligned} \tag{5.18}$$

*with $w \sim \mathcal{N}(0, \Sigma)$ is convex and tractable in the variables ($\eta$,$K$). Furthermore the hard input constraints (5.3) are satisfied under the control policy $u = \eta + Ke(w)$ if $||e(w)||_\infty \leq \varepsilon$. Here $K_i$ denotes the $i$-th row of $K$.*

*Proof.* The objective function is a sum of terms of the form $\mathbf{E}|q_{jk}^T x_k|$ or $\mathbf{E}|r_{jk}^T u_k|$, where $q_{jk}$, $r_{jk}$ denote the $j$-th rows of $Q_k$, $R_k$ respectively. Denote also

$$\mathcal{B}_k = [A^{k-1}B, \ldots, B, 0, \ldots, 0], \quad \mathcal{C}_k = [A^{k-1}, \ldots, I, 0, \ldots, 0]F,$$

where $\Sigma = FF^T$, and observe that

$$\begin{aligned}
q_{jk}^T x_k &= q_{jk}^T (A^k x_0 + \mathcal{B}_k u + \mathcal{C}_k \tilde{w}) \\
&= q_{jk}^T A^k x_0 + q_{jk}^T \mathcal{B}_k \eta + q_{jk}^T (\mathcal{C}_k + \mathcal{B}_k KF)\tilde{w}
\end{aligned}$$

with $\tilde{w} \sim \mathcal{N}(0, I)$. It is clear that $q_{jk}^T x_k$ is Gaussian with the expectation

$$\mu(\eta, k) = \mathbf{E}(q_{jk}^T x_k) = q_{jk}^T A^k x_0 + q_{jk}^T \mathcal{B}_k \eta, \tag{5.19}$$

and standard deviation

$$\sigma(\eta, k) = ||q_{jk}^T (\mathcal{C}_k + \mathcal{B}_k KF)||_2 = ||\mathcal{C}_k^T q_{jk} + (F^T \otimes q_{jk}^T \mathcal{B}_k) Sk||_2, \tag{5.20}$$

where $Sk = \text{vec}(K)$ with $S$ being a certain matrix of zeros and ones, and $k$ containing only the nonzero elements of $K$. Similarly

$$r_{jk}^T u_k = r_{jk}^T v_k \eta + r_{jk}^T v_k KF\tilde{w},$$

where $v_k$ is a vector that selects $k$-th block row of the size $m$. Consequently, the expectation and standard deviation become

$$\mu(\eta, k) = r_{jk}^T v_k \eta, \quad \sigma(\eta, k) = ||(F^T \otimes r_{jk}^T v_k) Sk||_2. \tag{5.21}$$

Application of Lemma 2, in the proof of which the gradient and Hessian were computed, now completes the convexity and tractability part of the proof. Satisfaction of the input constraints follows immediately from the definition of the induced infinity norm and from the assumption that $||e(w)||_\infty \leq \varepsilon$. $\square$

### 5.3.2 Bound on suboptimality

In this section we provide a bound on the suboptimality in (5.6) (with the same constraints on $\eta$, $K$ as in (5.18)) of the solution to the relaxed problem problem (5.18). The idea is to bound the difference of the costs under the policies $u = \eta + Kw$ and $u = \eta + Ke(w)$ for given $\eta$, $K$, which in effect bounds the difference of the respective optima. For ease of notation, the result is derived with time invariant weights, i.e. $Q_k := Q$, $R_k := R$ (and thus $q_{jk} := q_j$, $r_{jk} := r_j$) for all $k$, but generalizes immediately to the time varying case.

**Lemma 3.** *The cost $J_e$ incurred under the policy $u = \eta + Ke(w)$ and the cost $J_w$ incurred under the policy $u = \eta + Kw$ differ not more then*

$$(n_q(N+1)||Q||_\infty||\mathcal{B}_N||_\infty + n_r N||R||_\infty)\mathbf{E}||e(w) - w||_\infty||K||_\infty \tag{5.22}$$

*Proof.* We have

$$\begin{aligned} |J_e - J_w| \leq & \sum_{k=0}^{N}\sum_{j=1}^{n_q} |\mathbf{E}(|q_j^T x_k^e| - |q_j^T x_k^w|)| \\ & + \sum_{k=0}^{N-1}\sum_{j}^{n_r} |\mathbf{E}(|r_j^T u_k^e| - |r_j^T u_k^w|)|. \end{aligned} \tag{5.23}$$

Next, by Jensen's inequality,

$$\begin{aligned} |\mathbf{E}(|q_j^T x_k^e| - |q_j^T x_k^w|)| &\leq \mathbf{E}\,\big||q_j^T x_k^e| - |q_j^T x_k^w|\big| \\ &\leq \mathbf{E}(|q_j^T x_k^e - q_j^T x_k^w|) = \mathbf{E}|q_j^T \mathcal{B}_k K(e(w) - w)|, \end{aligned} \tag{5.24}$$

where

$$x_k^e = A^k x_0 + \mathcal{B}_k\eta + \mathcal{B}_k K e(w) + \mathcal{C}_k w, \quad x_k^w = A^k x_0 + \mathcal{B}_k\eta + \mathcal{B}_k K w + \mathcal{C}_k w.$$

Furthermore

$$\begin{aligned} \mathbf{E}|q_j^T \mathcal{B}_k K(e(w) - w)| &\leq ||q_j^T \mathcal{B}_k K||_\infty \mathbf{E}||e(w) - w||_\infty \\ &\leq ||q_j^T \mathcal{B}_k||_\infty ||K||_\infty \mathbf{E}||e(w) - w||_\infty \\ &\leq ||Q||_\infty ||\mathcal{B}_N||_\infty ||K||_\infty \mathbf{E}||e(w) - w||_\infty. \end{aligned} \tag{5.25}$$

Similar procedure can be carried out for control inputs to yield

$$|\mathbf{E}(|r_j^T u_k^e| - |r_j^T u_k^w|)| \le ||R||_\infty ||K||_\infty \mathbf{E}||e(w) - w||_\infty.$$

Summing up all terms in (5.23) now leads to the desired result

$$|J_e - J_w| \le (n_q(N+1)||Q||_\infty ||\mathcal{B}_N||_\infty + n_r N ||R||_\infty)\mathbf{E}||e(w) - w||_\infty ||K||_\infty,$$

which completes the proof. □

Now it is rather straightforward to derive the suboptimality bound. Denote $J_e^*$ the optimal value of (5.6) and the corresponding minimizer $K_e^*$, $\eta_e^*$. Denote also $J_w^*$ the optimal value of (5.18) and the corresponding optimal solution $K_w^*$, $\eta_w^*$. Finally denote $J_e$ the cost $J$ under the control policy $u = \eta_w^* + K_w^* e(w)$ and $J_w$ the cost $J$ under the policy $u = \eta_e^* + K_e^* w$.

**Theorem 2.** *The solution $\eta_w^*$, $K_w^*$ of (5.18) is not more than*

$$\beta := 2(n_q(N+1)||Q||_\infty ||\mathcal{B}_N||_\infty + n_r N ||R||_\infty)\mathbf{E}||e(w) - w||_\infty \frac{U_{\max}}{\varepsilon} \quad (5.26)$$

*suboptimal in (5.6).*

*Proof.* It follows from Lemma 3 that

$$|J_e - J_w^*| \le \frac{\beta}{2}, \quad |J_w - J_e^*| \le \frac{\beta}{2}$$

since $||K_e^*||_\infty \le U_{\max}/\varepsilon$, $||K_w^*||_\infty \le U_{\max}/\varepsilon$ because of the constraint on $K$ and $\eta$ in both optimization problems:

$$|\eta_i| + \varepsilon ||K_i||_\infty \le U_{\max},\ i = 1, \ldots, mN$$

implies $||K||_\infty \le U_{\max}/\varepsilon$.

Now since $J_e^* \le J_e$ and $J_w^* \le J_w$ the bound immediately follows

$$0 \le J_e - J_e^* \le J_e - J_w^* + J_w - J_e^* = |J_e - J_w^* + J_w - J_e^*| \le \beta,$$

which completes the proof. □

The term $\mathbf{E}||e(w) - w||_\infty$ in (5.26) can be computed to virtually arbitrary precision by means of a Monte Carlo simulation. The bound also provides an intuitively obvious guide to selecting the function $e(w)$ in such a way that $e(w)$ and $w$ do not differ very much with high probability. For instance with the choice of $e(w)$ as the elementwise saturation $e_i(w_i) = \mathrm{sat}_r(w_i)$ with $r \gtrsim 4\sqrt{\rho(\Sigma)}$ it is highly likely that the bound will be close to zero and, consequently, the solution to the relaxed problem will be almost optimal in the original one. Note also that this fairly crude bound can be significantly improved by terminating one inequality earlier in (5.25) at the cost of a slightly more complicated expression.

### 5.3.3 Receding horizon stability

In this section we provide a slight generalization and a much simplified proof of a result that already appeared in [10].

**Theorem 3.** *Let $u_k$, $w_k$ be two stochastic processes defined on the same probabilistic space with $||u_k||_\infty \leq U_{\max}$ a.s. and $\sup_{i,j} ||\mathbf{E}\{w_i w_j^T\}|| < \infty$. The state of the system $x_{k+1} = Ax_k + Bu_k + w_k$ then stays mean-square bounded (i.e. $\sup_k \mathbf{E}||x_k||_2^2 < \infty$) provided that $\mathbf{E}||x_0||_2^2 < \infty$ and $\rho(A) < 1$.*

*Proof.* $\mathbf{E}||x_k||_2^2 = \operatorname{tr}(\mathbf{E}\{x_k x_k^T\})$ and consequently it suffices to show that $\mathbf{E}\{x_k x_k^T\}$ is bounded in any norm because of the norm equivalence on finite dimensional vector spaces and the fact that $\operatorname{tr}(\cdot)$ coincides with the nuclear norm on the space of positive semidefinite matrices. The proof proceeds by direct evaluation:

$$\begin{aligned}
\mathbf{E}(x_k x_k^T) &= \mathbf{E}\{(A^k x_0 + \mathcal{B}_k U_k + \mathcal{C}_k W_k)(A^k x_0 + \mathcal{B}_k U_k + \mathcal{C}_k W_k)^T\} \\
&= A^k P_0 (A^k)^T + A^k \mathbf{E}\{x_0 U_k^T\}\mathcal{B}_k^T + \mathcal{B}_k \mathbf{E}\{U_k x_0^T\}(A^k)^T \\
&+ \mathcal{B}_k \mathbf{E}\{U_k U_k^T\}\mathcal{B}_k^T + \mathcal{B}_k \mathbf{E}\{U_k W_k^T\}\mathcal{C}_k^T + \mathcal{C}_k \mathbf{E}\{W_k U_k^T\}\mathcal{B}_k^T \\
&+ \mathcal{C}_k \mathbf{E}\{W_k W_k^T\}\mathcal{C}_k^T,
\end{aligned} \tag{5.27}$$

where

$$\begin{aligned}
U_k &= [u_0^T, \ldots, u_{k-1}^T]^T, \quad W_k = [w_0^T, \ldots, w_{k-1}^T]^T, \\
\mathcal{B}_k &= \left[A^{k-1}B, \ldots, B\right], \quad \mathcal{C}_k = \left[A^{k-1}, \ldots, I\right].
\end{aligned}$$

The boundedness of the first term is obvious, the boundedness of the second and third terms follows from the fact that $||\mathbf{E}\{x_0 U_k^T\}||_2 \leq U_{\max}\sqrt{mk\mathbf{E}||x_0||_2^2}$ (this follows directly by Jensen's and Cauchy-Schwarz inequalities). The boundedness of $\mathcal{B}_k$ is obvious by the assumption that $\rho(A) < 1$, and therefore the second and third terms actually go to zero. Consider now any $\Delta < \infty$ bounded family of matrices $M_{rq}$, i.e. $||M_{rq}|| \leq \Delta$ for all $r$, $q$. For such a family we have

$$\begin{aligned}
\left|\left|\sum_{i=0}^{k-1}\sum_{j=0}^{k-1} A^i M_{rq} A^j\right|\right| &\leq \sum_{i=0}^{k-1}\sum_{j=0}^{k-1} ||A^i||||M_{rq}||||A^j|| \\
&\leq \Delta \sum_{i=0}^{k-1}\sum_{j=0}^{k-1} ||A^i||||A^j||.
\end{aligned} \tag{5.28}$$

The first term in (5.28) is therefore bounded since the last series is convergent by the assumption that $\rho(A) < 1$ (for instance by taking the Jordan form of the matrices and choosing a suitable norm, e.g. the Frobenius norm). Here $||\cdot||$ can be any submultiplicative norm. The theorem then follows since the last four terms in (5.27) can be casted in the stated form with $r = k - i - 1$, $q = k - j - 1$ and $M_{rq}$ componentwise bounded (by Cauchy-Schwarz inequality and the assumptions on $u_k$, $w_k$) and hence $||\cdot||$ bounded due to the norm equivalence. $\square$

**Corollary 1.** *The receding horizon implementation of the control policy defined by solving the optimization problem (5.18) every $N_c \leq N$ steps and applying the first $N_c$ control inputs generated by the policy $u = \eta + Ke(w)$ renders the state $x_k$ mean-square bounded provided that $\rho(A) < 1$.*

*Proof.* Follows directly from Theorem 3 since the constraints in (5.18) ensure that the inputs stay bounded. $\square$

In the case of $\rho(A) = 1$ with the deterministic part of the system (5.2) Lyapunov stable, the sole assumption of bounded control inputs is insufficient, and another constraint must be embedded into (5.18) in order to ensure the mean-square boundedness of the state. See [11] for details.

## 5.4 Numerical examples

We present two numerical examples that compares our method to other control strategies. With the gradient and Hessian on hand, the problem (5.18) can be solved by a nonlinear solver with guaranteed convergence because of convexity or by a general purpose convex solver. For our small scale examples we managed with the Matlab nonlinear solver implemented in the FMINCON function with the 'interior-point' option as well as with a custom interior-point solver. Nondifferentiability of the objective is not a problem in our case since the the optimization path and the solution itself lie outside the nondifferentiable region. If this were not the case, which can happen if the penalty on control effort is large leading to zero mean and zero variance of a particular control input, various techniques for nodifferentiable convex optimization can be employed.
In the first example we consider a fixed horizon stochastic control problem. For the the system matrices and the noise covariance matrix we chose

$$A = \begin{bmatrix} 1 & -0.4 \\ 0.1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}, \; \Sigma = I \otimes \begin{bmatrix} 8 & 5 \\ 5 & 6 \end{bmatrix}$$

with $w_k$ zero-mean jointly Gaussian. We set $Q = I$, $R = 0.1I$, and the input constraints to $U_{\max} = 30$. The optimization horizon is $T = 12$, the initial state $x_0 = [1, -1]^T$. The function $e(w)$ was chosen as suggested above to be the elementwise saturation that saturates the disturbances to

$4\sqrt{\rho(\Sigma)} = 13.9$. We compared our method (with $N_c = N = T$) with the standard certainty equivalent MPC ($N_c = 1$, $N = T$) and with the shrinking horizon CE-MPC ($N_c = 1$, $N(k) = T - k$, $k = 0, \dots, T-1$). Furthermore, we tried out the proposed method with $K = 0$ against the certainty equivalent open loop control (i.e. CE-MPC with $N_c = N = T$). For the sake of completeness we tried out our method in the shrinking horizon mode with $N_c = 2, N(k) = T - k$ as well. The respective objective functions were evaluated using 2000 Monte Carlo runs. The results are summarized in Table 5.1, which shows that our method (without shrinking) outperforms the others by a significant margin except perhaps for SH-MPC where the difference is smaller and, naturally, our method in the shrinking horizon regime. On the other hand, unlike with MPC strategies, there is no need for online optimization with our method in this setting. It is also worth noting that our method with $K = 0$ (i.e. an open loop policy) slightly outperforms the certainty equivalent open loop control, which is in contrast with the quadratic cost case where this strategy is optimal in the class of open loop policies. Figure 5.1 shows histograms of the proposed policy and the two MPC policies. Finally, we evaluate the bound (5.26) which yields $\beta = 0.005$ showing that the solution found by (5.18) is in this case basically optimal in (5.6).

Table 5.1: Comparison of control policies over the optimization horizon $T = 12$.

| Policy | SH-(5.18) | (5.18) | SH-MPC | MPC | (5.18), $K = 0$ | OL |
|---|---|---|---|---|---|---|
| $J$ | 86.8 | 92.1 | 98.3 | 119.2 | 140.4 | 143.9 |

Our second example compares the proposed method with the certainty equivalent MPC in a receding horizon regime. In this example we consider the respective matrices

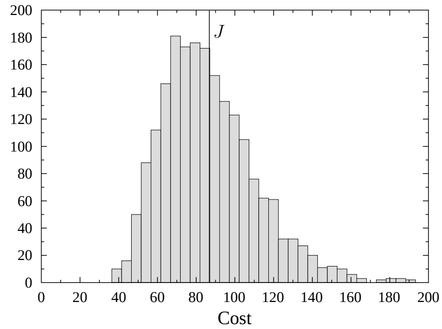
$$A = \begin{bmatrix} 1 & 1 \\ -0.5 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{E}\{w_i w_j^T\} = \begin{bmatrix} 8 & 5 \\ 5 & 6 \end{bmatrix} \delta_{ij}$$

with $w_k$ zero-mean Gaussian and independent, where $\delta_{ij}$ denotes the Kronecker delta. The weighting matrices were set to $Q = I$ and $R = 0.1I$, the input constraints to $U_{\max} = 10$, and the initial state to $x_0 = [1, -1]^T$. We compared our control policy with $N = 12$, $N_c = 4$ against CE-MPC with $N = 12$, $N_c = 1$ in a receding horizon fashion over the simulation time $T = 100$. Again, we used the 4-sigma rule to get $\varepsilon = 13.9$. Figure 5.2 shows the accumulation of the cost over the simulation time, while Figure 5.3 depicts the evolution of the state's 2-norm-square expectation suggesting its boundedness, which was to be expected since $\rho(A) = \sqrt{2}/2$. 100 Monte

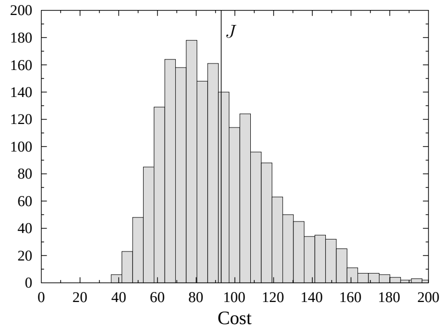
Carlo runs were used to evaluate the expectations in the costs.

## 5.5 Conclusion

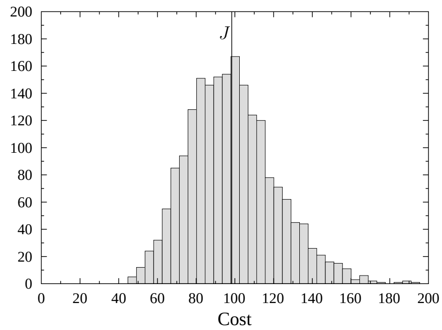
In spite of the natural requirement of bounded control inputs, surprisingly little research effort in the field tackles this problem directly in the presence of unbounded stochastic disturbances.
In this article, we dealt exclusively with the expectation of the 1-norm stochastic control problem for which we developed an approximate solution technique ensuring bounded control inputs in the presence of Gaussian disturbances. Moreover, we constructed a suboptimality bound of our method in a certain class of nonlinear disturbance feedback control policies. Finally, we provided a simple proof of receding horizon stability of the proposed policy, and demonstrated our results by means of two numerical examples.
Since Gaussian random variables are assumed, it is straightforward to include individual chance constraints leading to additional second-order-cone constraints. If joint chance constraints were of interest, it is possible to adopt the methodology of [5] resulting in a non-convex problem, which can, however, be solved by simple sequential convex programming, providing promising results.
Furthermore, the question of the mean-square boundedness of Lyapunov unstable systems with the system matrix of spectral radius one remains, at least to our knowledge, open. Finally, it would be interesting to develop a tractable way of obtaining a global lower bound on the optimal value of the infinite horizon 1-norm stochastic control problem using the approach of [21].



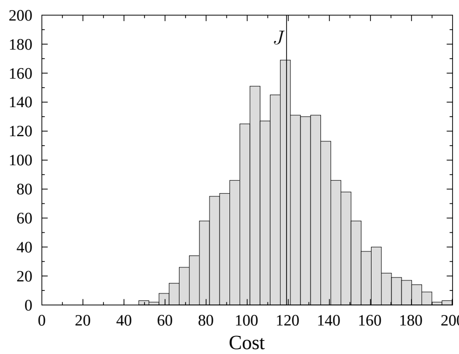

(a) Proposed policy (5.18) in the shrinking horizon regime with $N_c = 2$, $N(k) = T - k$




(b) Proposed policy (5.18) with $N_c = N = T$




(c) Shrinking horizon CE-MPC ($N_c = 1$, $N(k) = T - k$)




(d) CE-MPC ($N_c = 1$, $N = T$)

Figure 5.1: Histograms of the costs of different control policies over 2000 Monte Carlo runs on the optimization horizon $T = 12$.

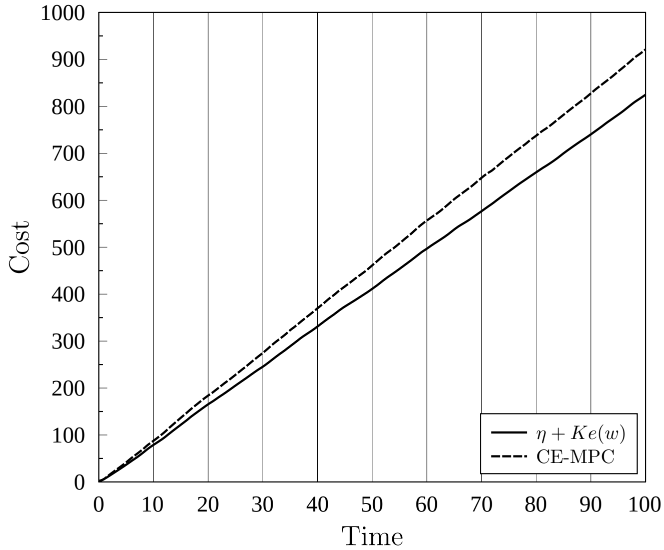



Figure 5.2: Comparison of costs over the simulation time $T = 100$ in a receding horizon regime with $N = 12$ and $N_c = 4$ for our policy and $N_c = 1$ for CE-MPC. Final costs are 824.7 for our control scheme and 921.1 for CE-MPC.

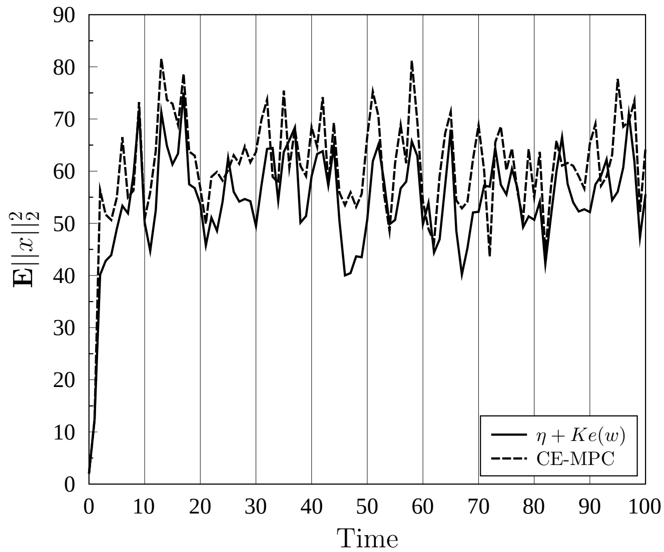



Figure 5.3: Evolution of $\mathbf{E}||x||_2^2$ under our receding horizon control policy with $N = 12$, $N_c = 4$ and CE-MPC control policy with $N = 12$, $N_c = 1$.

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