EVROPSKÁ UNIE

MINISTERSTVO ŠKOLSTVÍ, MLÁDEŽE A TĚLOVÝCHOVY

OP Vzdělávání pro konkurenceschopnost

VYSOKÉ UČENÍ TECHNICKÉ V BRNĚ

INVESTICE DO ROZVOJE VZDĚLÁVÁNÍ

Signal Processing

Random Variables -Random Signals, Digital Filters, Kalman Filter, Vold-Kalman Order Tracking Filter, Analytic Signals and Hilbert Transform

Harmonic Signal Modulation -Amplitude and Phase Demodulation

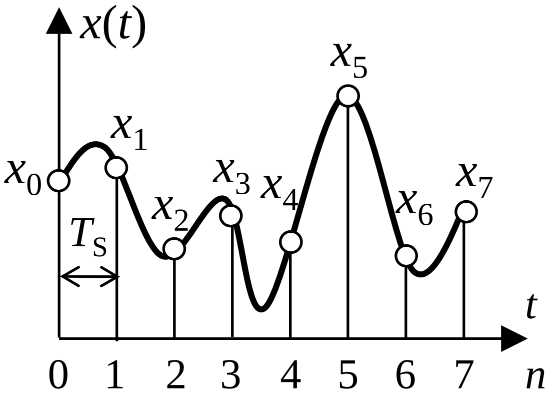
Jiří Tůma

VŠB – Technical University of Ostrava

Tato prezentace je spolufinancována Evropským sociálním fondem a státním rozpočtem České republiky.

Signals with continuous and discrete time, signal types

The signal $x(t)$ is a real or complex function of continuous time t. The other definition points to the fact that signal contains information. Sampling of a signal produces a time series which is a sequence of samples in the discrete time n. The sequence of samples may be denoted either as an indexed variable or as a function of an integer number $x(n)$.



Sampling may be considered as a mapping

$$\text{Sampling}: \quad x(t) \quad \rightarrow \quad [x_0, x_1, x_2, \dots]^T$$

The time continuous signal is related to the time series and can be substituted by the following way

$$\sum_n x(t)\delta(t - nT_S) = \sum_n x_n \delta(t - nT_S) \quad \rightarrow \quad x(t)$$

where T_S is a sampling interval for the uniformly sampled data. The sampling frequency (rate) f_S is the reciprocal value of the sampling interval.

Signal types

Deterministic				Random (stochastic)		
Periodic		Nonperiodic		Stationary		Nonstationary
Sinusoidal	Complex periodic (harmonic)	Almost periodic	Transient	Ergodic	Non-ergodic	Special classification

Deterministic signals are defined as a function of time while random signals can be defined in terms of statistical properties.

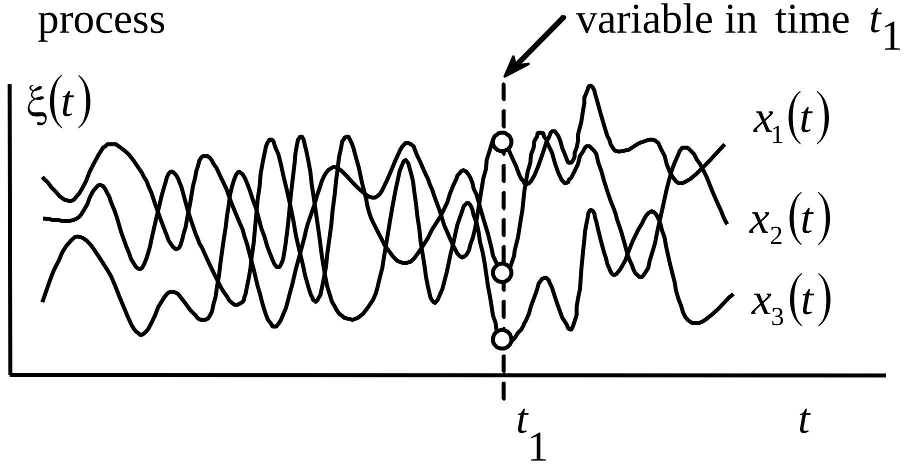
Random variables, random signals

Names of random variables and processes: $\xi, \xi(t), \varepsilon, \varepsilon(t), \ldots$

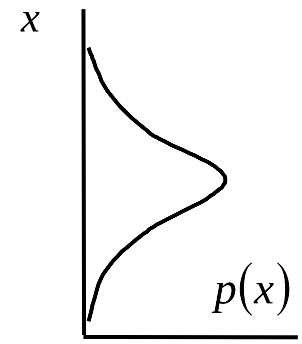
Realization of random variables and processes: $x, x(t), y, y(t), \ldots$

Realization of the random process

Realization of the random variable in time t_1



Independent variable



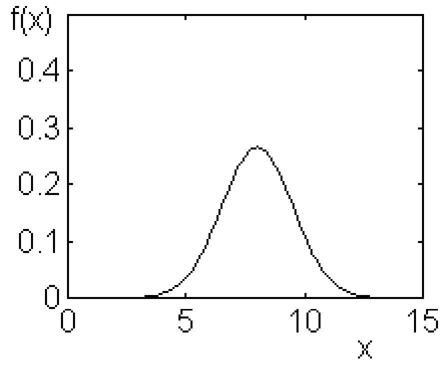
Probability density function

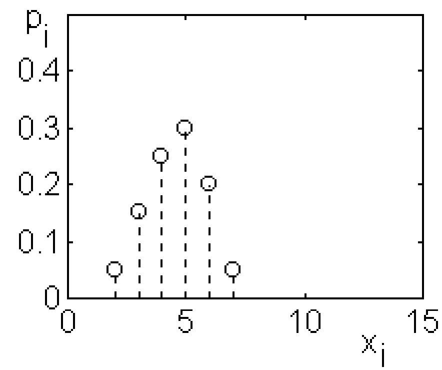
Probability that a random variable ξ belongs to the interval of values greater than x and less than $x+\Delta x$ is proportional to the interval of the length Δx

$$P\{x < \xi \le x + \Delta x\} = p(x)\Delta x$$

The coefficient of proportionality is denoted as a probability density function (pdf).

Probability density function



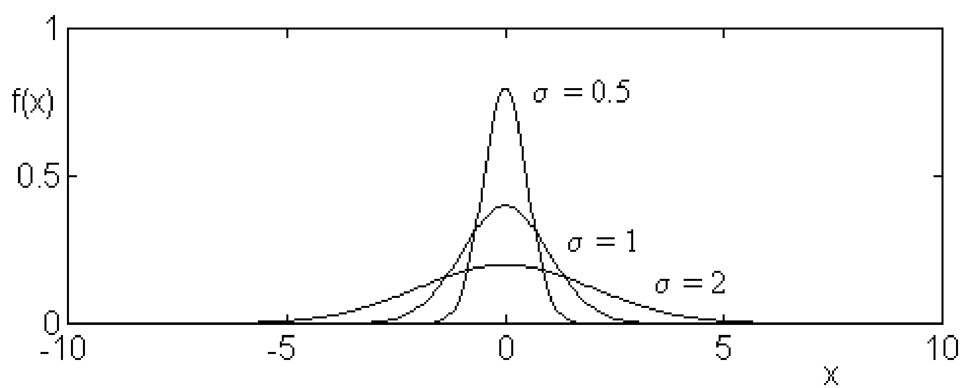


Continuous random variable < > Discrete random variable

Probability density function for
the normal (Gaussian) distribution

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right),$$

$$-\infty < x < +\infty$$



Gaussian function (bell curve)

Mean value, variance and correlation function

For a random variable ξ, it is introduced

Mean value ………….. $\mu = \mathrm{E}\{\xi\} = \int_{-\infty}^{+\infty} x\,p(x)\,\mathrm{d}\,x$ Expected value (moment in physics)

Centered variable ……. $\Delta x = x - \mathrm{E}\{x\}, \quad (\mathrm{E}\{\Delta x\} = 0)$

The *n*th central moment $M_k = \mathrm{E}\{\xi^k\} = \int_{-\infty}^{+\infty} x^k\, p(x)\,\mathrm{d}\,x$

The *n*th central moment about a mean value μ $m_k = \mathrm{E}\{(\xi-\mu)^k\} = \int_{-\infty}^{+\infty} (x-\mu)^k\, p(x)\,\mathrm{d}\,x$

Variance ……………..…….. $\mathrm{D}\{\xi\} = \sigma^2 = m_2$

Crest factor ……..……. $\text{Crest factor} = (\text{Peak value})/\sigma$

For a random signal (process) $\xi(t)$, it is defined

Correlation function $R_{xx}(t_1, t_2) = \mathrm{E}\{\xi_1(t_1)\xi_2(t_2)\} =$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x_1(t_1)x_2(t_2)\,p(x_1, x_2, t_1, t_2)\,\mathrm{d}\,x_1\,\mathrm{d}\,x_2$$

The two-dimensional probability density function $p(x_1, x_2, t_1, t_2)$

Covariance

The covariance between two real-valued random variables X and Y with finite second moments is

$$\operatorname{cov}(X,Y) = \mathrm{E}\{(X - E\{X\})(Y - E\{Y\})\}$$

The covariance between random vectors X and Y of dimension mx1 and nx1, respectively

$$\operatorname{cov}(X,Y) = \mathrm{E}\left\{(X - E\{X\})(Y - E\{Y\})^T\right\} = E\left\{XY^T\right\} - E\{X\}E\{Y\}^T$$

Properties

Let X, and Y be real-valued random variables and a, b be constant ("constant" in this context means non-random), then it holds

$$\operatorname{cov}(X,a) = 0$$

$$\operatorname{cov}(X,X) = \operatorname{var}(X)$$

$$\operatorname{cov}(X,Y) = \operatorname{cov}(Y,X)$$

$$\operatorname{cov}(aX,bY) = ab\operatorname{cov}(X,Y)$$

$$\operatorname{cov}(X+a,Y+b) = \operatorname{cov}(X,Y)$$

If X and Y are independent, then their covariance is zero. It follows

$$\mathrm{E}\{XY\} = \mathrm{E}\{X\}\mathrm{E}\{Y\}$$

Stationary and ergodic signals I

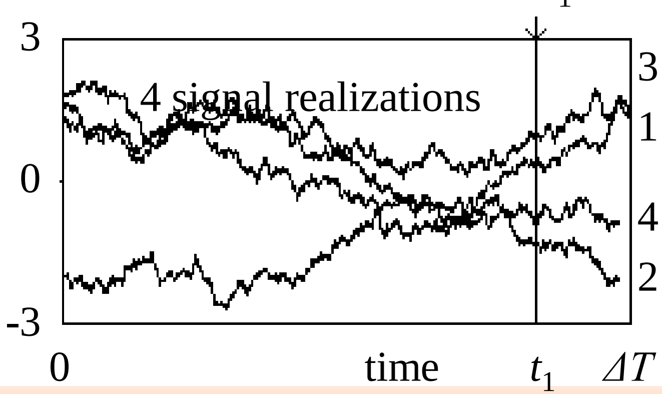
A stationary signal (or strict(ly) or strong(ly) stationary signal) is a stochastic process whose joint probability distribution does not change when shifted in time or space. As a result, parameters such as the mean and variance, if they exist, also do not change over time.

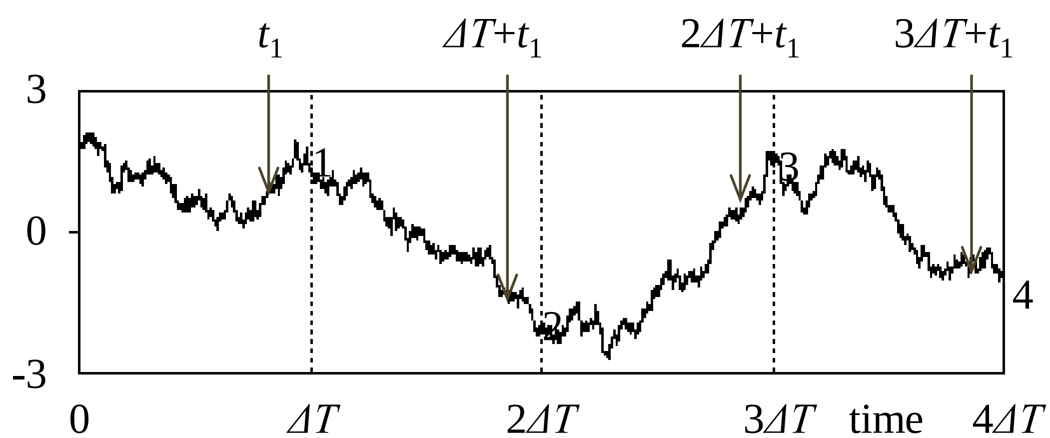
An ergodic process is one which conforms to the ergodic theorem. The theorem allows the time average of a conforming process to equal the ensemble average. In practice this means that statistical sampling can be performed at one instant across a group of identical processes or sampled over time on a single process with no change in the measured result.

For example the parameters (mean and variance) can be computed from values corresponding to random signal realizations in time t_1.

If the realizations are sections of the long record and ΔT tends to zero or to the interval, which is small enough, them the values across a group of identical processes can be replaced by the samples of the time record.

Random variables in time t_1





Stationary and ergodic signals II

The basic properties of stationary continuous signals $x(t)$ is as follows

$$p(x_1, t_1) = p(x_1)$$
$$p(x_1, x_2, t_1, t_2) = p(x_1, x_2, t_1 - t_2)$$

$$R_{xx}(t_1, t_2) = R_{xx}(t_2 - t_1) = R_{xx}(\tau)$$

For ergodic signals, it is assumed that the mean value can be replaced by the time average

$$\bar{x} = \lim_{T \to +\infty} \frac{1}{T} \int_{-T/2}^{+T/2} x(t)\,\mathrm{d}t$$

$$s = \sqrt{\lim_{T \to +\infty} \frac{1}{T} \int_{-T/2}^{+T/2} (x(t) - \bar{x})^2\,\mathrm{d}t}$$

Root Mean Square = RMS

The correlation function of ergodic signals depends only on the lag $R_{xx}(\tau) = \lim_{T \to +\infty} \frac{1}{T} \int_{-T/2}^{+T/2} x(t) x(t-\tau)\,\mathrm{d}t$

For discrete ergodic random signals ($x(i)$, i = 0, 1, 2, …), the formulas take the form

Mean value $\bar{x} = \frac{1}{N} \sum_{i=0}^{N-1} x(i)$

Standard deviation $s = \sqrt{\frac{1}{N} \sum_{i=0}^{N-1} (x(i) - \bar{x})^2}$

Auto-correlation

$$R_{xx}(\tau) = \frac{1}{N-\tau} \sum_{i=0}^{N-\tau-1} x(i) x(i+\tau), \quad \tau = 0, 1, 2, \ldots, N-2$$

Cross-correlation

$$R_{xy}(\tau) = \frac{1}{N-\tau} \sum_{i=0}^{N-\tau-1} x(i) y(i+\tau), \quad \tau = 0, 1, 2, \ldots, N-2$$

Mean value and standard deviation (RMS) of a sine signal

If the signal is harmonic (sinusoidal) than we can calculate

Sinusoidal signal …… $x(t) = A\cos\left(\frac{2\pi}{T}t\right)$

Mean value ………… $\bar{x} = \frac{1}{T}\int_0^T x(t)\,\mathrm{d}t = 0$

The computation of RMS for the sinusoidal signal

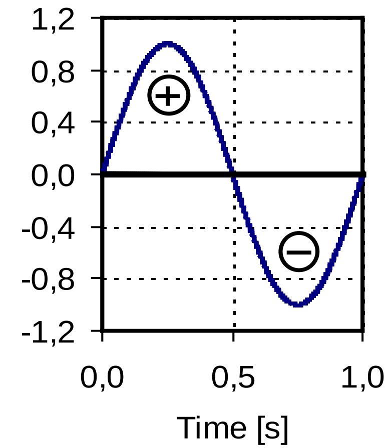
$$\sigma^2 = \frac{1}{T}\int_0^T (x(t))^2\,\mathrm{d}t = \frac{1}{T}\int_0^T \left(A\cos\left(\frac{2\pi}{T}t\right)\right)^2\mathrm{d}t$$

$$= \frac{1}{T}\int_0^T \frac{A^2}{2}\left(1+\cos\left(2\frac{2\pi}{T}t\right)\right)\mathrm{d}t = \frac{A^2}{2}$$

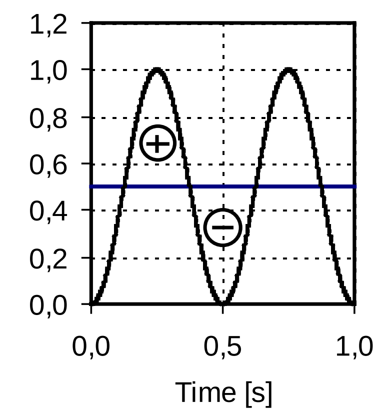
results in the formula $\sigma = \frac{A}{\sqrt{2}}$

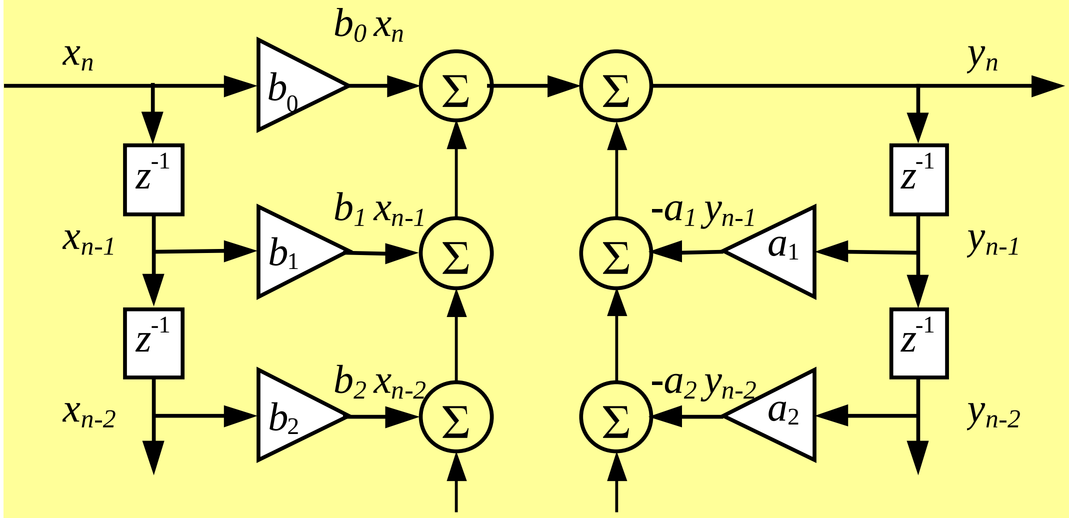
The value of RMS is approximately equal to 70% of the harmonic signal amplitude. The amplitude of this signal is the 1.4-multiple of RMS.

$x(t)$



$(x(t))^2$

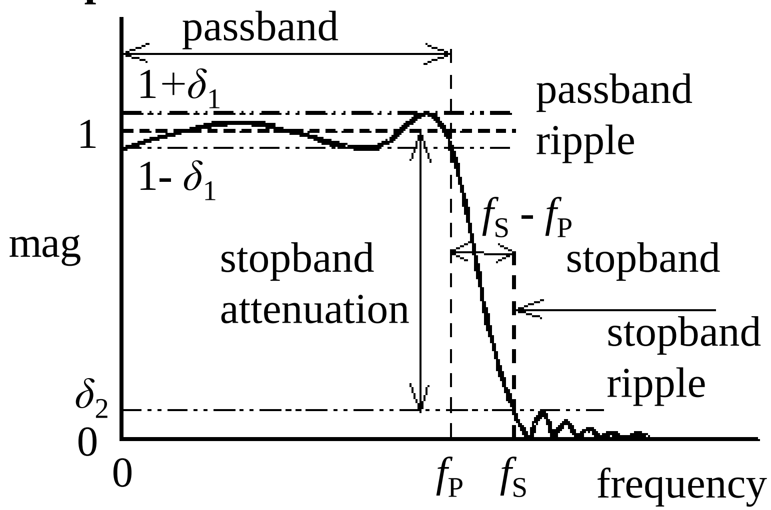




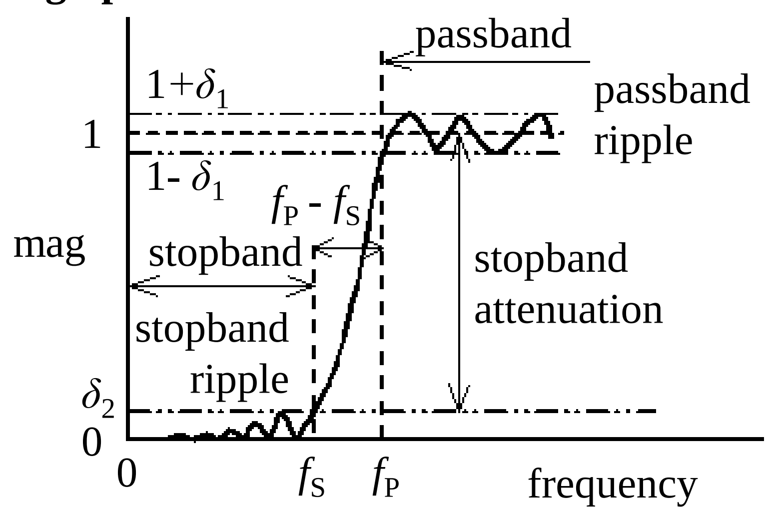
DIGITAL FILTERS

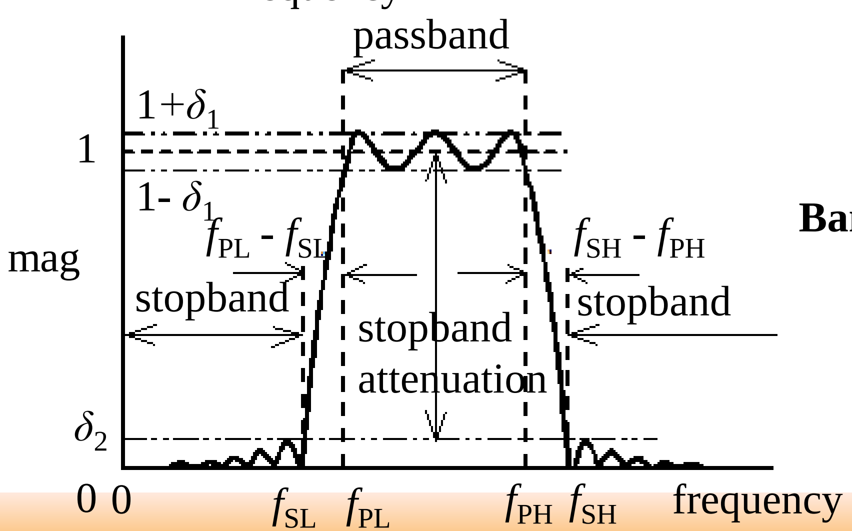
Filter frequency response

Low pass filter



High pass filter





Band pass filter

Z-transform of a sequence of samples

Let $x_0, x_1, x_2, x_3, \ldots$ be a sequence of samples

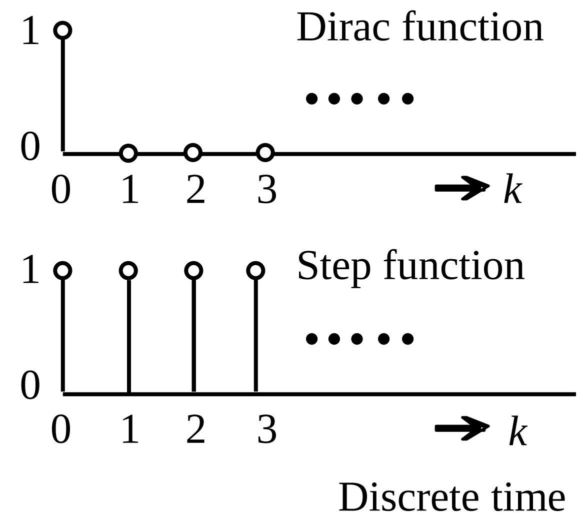
The Z-transform of the sample sequence is defined by

$$X(z) = Z\{x_n\} = x_0 + x_1 z^{-1} + x_2 z^{-2} + x_3 z^{-3} + \ldots = \sum_{n=0}^{+\infty} x_n z^{-n}$$

where z is a complex variable.

For the k-step delay it is valid $Z\{x_{n-k}\} = z^{-k} Z\{x_n\}$

Examples of the Z-transform of some signals



Dirac sequence: 1, 0, 0, 0, …

$$X(z) = Z\{\delta_n\} = 1 + 0\,z^{-1} + 0\,z^{-2} + \ldots = 1$$

Step function: 1, 1, 1, 1, …

$$X(z) = Z\{s_n\} = 1 + z^{-1} + z^{-2} + \ldots = \frac{1}{1-z^{-1}} = \frac{z}{z-1}$$

k-1

Upsampling by k: x_0, ~~0, 0, …,~~ x_1, 0, 0, …, x_2, 0, 0, …

If $X(z) = x_0 + x_1 z^{-1} + x_2 z^{-2} + \ldots$ then $X_k(z) = X(z^k)$

Convolution in the discrete time domain

Let the following sequences be denoted by

input samples $x_i,\ i = 0, 1, 2, 3, \ldots$

impulse response $h_i,\ i = 0, 1, 2, 3, \ldots$

output samples $y_i,\ i = 0, 1, 2, 3, \ldots$

The convolution of the input sequence with the impulse response is a formula

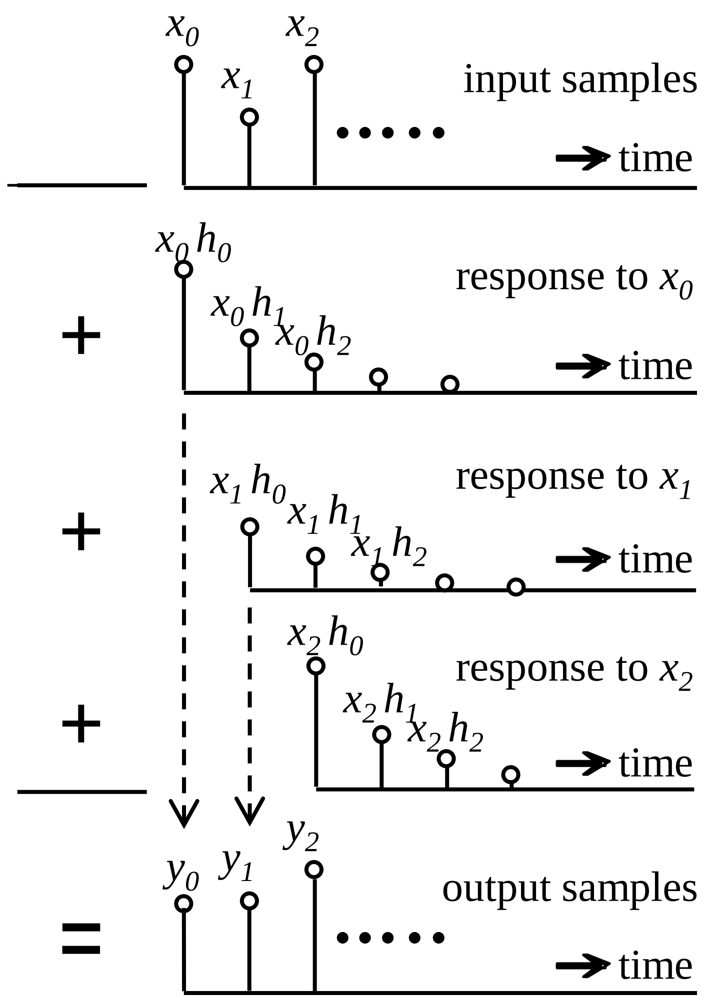
$$y_n = h_n \otimes x_n = \sum_{i=0}^{+\infty} h_i x_{n-i} = \sum_{i=-\infty}^{n} h_{n-i} x_i$$

The Z-transform of the convolution sum results in

$$Z\{y_n\} = Z\left\{\sum_{i=0}^{+\infty} h_i x_{n-i}\right\} = \sum_{n=0}^{+\infty}\sum_{i=-\infty}^{n} h_{n-i} x_i z^{-n} =$$

$$= \left(\sum_{n=0}^{+\infty} h_n z^{-n}\right)\left(\sum_{n=0}^{+\infty} x_n z^{-n}\right)$$

$$Y(z) = H(z)X(z)$$



Digital filter as the discrete-time linear systems

A digital filter can be considered as a discrete time linear system with one input and one output. The input sequence is designated by x_n, $n = 0,1,2, \dots$ while the output sequence by y_n, $n = 0,1,2, \dots$ The present input sample is $x(n)$ and the next output sample is $y(n)$ in the alternative notation.

Let a constant coefficient linear difference equation with zero initial conditions be given

$$y_n = -a_1 y_{n-1} - \dots - a_N y_{n-N} + b_0 x_n + b_1 x_{n-1} + \dots + b_M x_{n-M}$$

or

$$y_n + a_1 y_{n-1} + \dots + a_{N-1} y_{n-N+1} + a_N y_{n-N} = b_0 x_n + b_1 x_{n-1} + \dots + b_{M-1} x_{n-M+1} + b_M x_{n-M}$$

The Z-transform of the difference equation of the order N gives

$$Y(z)\left(1 + a_1 z^{-1} + \dots + a_{N-1} z^{-N+1} + a_N z^{-N}\right) = X(z)\left(b_0 + b_1 z^{-1} + \dots + b_{M-1} z^{-M+1} + b_M z^{-M}\right)$$

The transfer function of the linear system in the Z-domain is in the form of two polynomials composed from the power of z^{-1}

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + \dots + b_{M-1} z^{-M+1} + b_M z^{-M}}{1 + a_1 z^{-1} + \dots + a_{N-1} z^{-N+1} + a_N z^{-N}}$$

A stable filter assures that every limited input signal produces a limited filter response. The zeros of the denominator polynomial $z^N + a_1 z^{N-1} + \dots + a_{N-1} z + a_N = 0$

have to fulfill $|z_i| < 1$. If the input squence is the Dirac function then the output sequence is called as the impulse response.

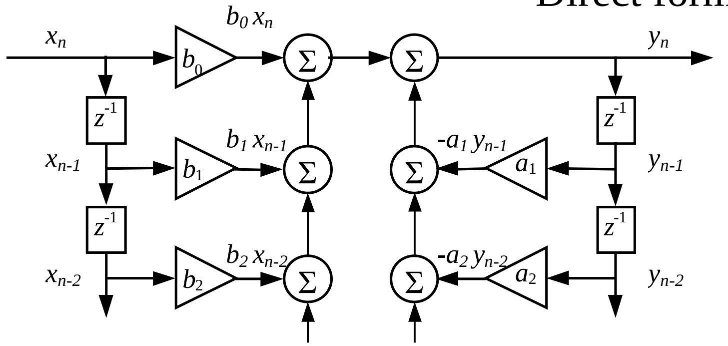
IIR versus FIR filters

IIR - Infinite Impulse Response

$$H(z)=\frac{Y(z)}{X(z)}=$$

$$=\frac{b_0+b_1z^{-1}+\ldots+b_{M-1}z^{-M+1}+b_Mz^{-M}}{1+a_1z^{-1}+\ldots+a_{N-1}z^{-N+1}+a_Nz^{-N}}$$

Direct form



IIR filter output of the order N

$$y_n=b_0x_n+b_1x_{n-1}+\ldots+b_Mx_{n-M}-$$

$$-a_1y_{n-1}-\ldots-a_Ny_{n-N}$$

Filter properties

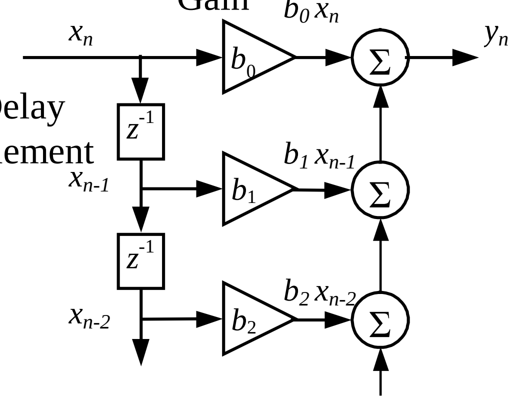
Positiveness – low order

Negativeness – filter can be unstable

FIR - Finite Impulse Response

$$H(z)=\frac{Y(z)}{X(z)}=$$

$$=b_0+b_1z^{-1}+\ldots+b_{M-1}z^{-M+1}+b_Mz^{-M}$$



FIR filter output of the order M

$$y_n=b_0x_n+b_1x_{n-1}+\ldots+b_{M-1}x_{n-M+1}+b_Mx_{n-M}$$

Filter properties

Positiveness – always stable

Negativeness – requier high order

Mapping of the s-plane imaginary axis onto the z-plane

The basic notation and assumptions are as follows

Sampling interval T_S, sampling frequency $f_S = 1/T_S$, sampling angular frequency $\omega_S = 2\pi/T_S$

One step delay $z^{-1} = \exp(-sT_S)$, $z^{-1} = \exp(-j\omega T_S)$ or $z = \exp(sT_S)$, $z = \exp(j\omega T_S)$

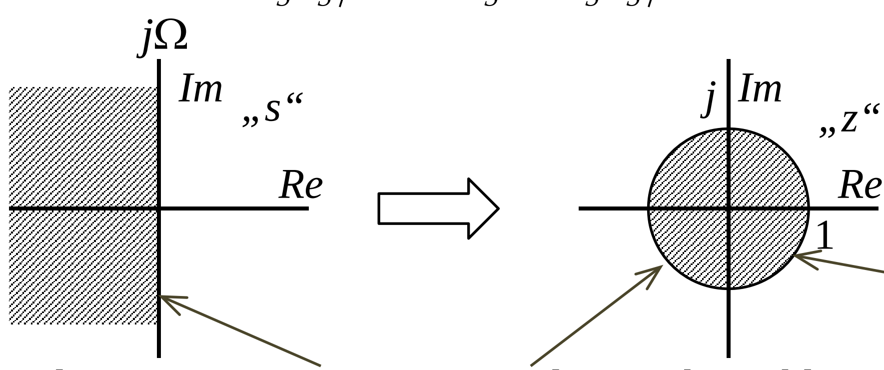
The Nyquist–Shannon sampling theorem is a fundamental result in the field signal processing. Shannon's version of the theorem states (see wikipedia):
If a function $x(t)$ contains no frequencies higher than f_m hertz ($\omega_S = 2\pi f_m$ radians per a second), it is completely determined by giving its ordinates at a series of points spaced $1/(2f_m)$ seconds apart.

The imaginary axis of the s-plane ranges from minus infinity to plus infinity $-\infty < \Omega < +\infty$

Sufficient frequency range of a sampled signal is in radians per a second or just in radians as follows

$$-\omega_S/2 \le \omega \le \omega_S/2 \qquad -\pi = -\omega_S T_S/2 \le \omega T_S \le \omega_S T_S/2 = \pi$$

Analog plane **Digital plane**



The unit circle is the locus of points in the complex plane, where

$$z = \exp(j\omega T_S)$$
$$-\infty < \omega < +\infty$$

Margin of regions for denominator zeros corresponding to the stable transfer functions

Digital filter frequency response

Let frequency properties of a filter be described by the transfer function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{1 + a_1 z^{-1} + \ldots + a_N z^{-N}}$$

Mapping of the *s*-plane imaginary axis onto the unit circle in the *z*-plane is ensured by substitution

$$z^{-1} = \exp(-j\omega T_S)$$

where T_S is the sampling frequency. This formula relates the variables z and ω. After substitution we obtain

$$H_C(j\omega) = H\left(e^{j\omega T_S}\right) = \frac{Y\left(e^{j\omega T_S}\right)}{X\left(e^{j\omega T_S}\right)} = \frac{b_0 + b_1 e^{-j\omega T_S} + \ldots + b_M e^{-jM\omega T_S}}{1 + a_1 e^{j\omega T_S} z^{-1} + \ldots + a_N e^{-jN\omega T_S}}$$

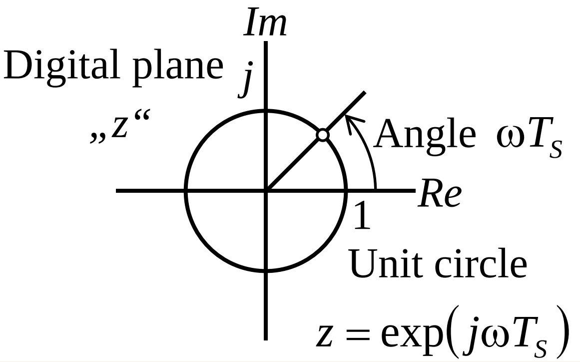
The value of $H(z)$ for z located at the unit circle is related to the value of the frequency response

The frequency range for computing the transfer function is as follows

in frequency ………… $-f_S/2 \le f \le f_S/2$

in angular frequency … $-\omega_S/2 \le \omega \le \omega_S/2$

in angle … $-\pi = -\omega_S T_S/2 \le \omega T_S \le \omega_S T_S/2 = \pi$



Computation of the magnitude and principal value of phase

Magnitude of the frequency response function

$$\left|H\left(e^{j\omega T_S}\right)\right| = \sqrt{\left(\mathrm{Re}\left\{H\left(e^{j\omega T_S}\right)\right\}\right)^2 + \left(\mathrm{Im}\left\{H\left(e^{j\omega T_S}\right)\right\}\right)^2}$$

The principal value concerns a logarithm of a complex non-zero number *z*. The principal value Log *z* is the logarithm whose imaginary part lies in the interval $(-\pi,\pi]$.

If the one-argument function *arctan* (*atan* in MATLAB) produces an angle in radians from the interval (-π/2, +π/2) that the principal value of phase from the interval (-π, +π] can be computed by the following formula

$$\mathrm{Arg}\left(H\left(e^{j\omega T_S}\right)\right) = \begin{cases} \arctan\left(\mathrm{Im}\{H\}/\mathrm{Re}\{H\}\right), & \text{for } \mathrm{Re}\{H\} > 0 \\ \pi + \arctan\left(\mathrm{Im}\{H\}/\mathrm{Re}\{H\}\right), & \text{for } \mathrm{Re}\{H\} < 0, \mathrm{Im}\{H\} \geq 0 \\ \arctan\left(\mathrm{Im}\{H\}/\mathrm{Re}\{H\}\right) - \pi, & \text{for } \mathrm{Re}\{H\} < 0, \mathrm{Im}\{H\} < 0 \\ +\pi/2, & \text{for } \mathrm{Re}\{H\} = 0, \mathrm{Im}\{H\} > 0 \\ -\pi/2, & \text{for } \mathrm{Re}\{H\} = 0, \mathrm{Im}\{H\} < 0 \\ \text{undefined}, & \text{for } \mathrm{Re}\{H\} = 0, \mathrm{Im}\{H\} = 0 \end{cases}$$

The MATLAB two-argument function *atan2* produces an angle in radians from the interval (-π, +π]

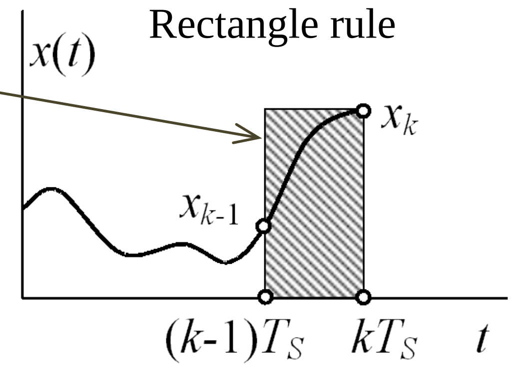
Approximation of the numerical integration

Mapping the plane „*s*“ onto the plane „*z*“ depends on algorithm, which is used to approximate a definite integral. We deals with the algorithms, which are based on the rectangle and trapezoidal rule.

$$y_n = y_{n-1} + T_S x_n$$

$$Y(z)\left(1 - z^{-1}\right) = T_S X(z) \quad \Rightarrow \quad \frac{Y(z)}{X(z)} = \frac{T_S}{1 - z^{-1}} \approx \frac{1}{s}$$

$$\boxed{s \approx \frac{1}{T_S}\left(1 - z^{-1}\right)} \qquad \boxed{z \approx \frac{1}{1 - T_S s}}$$



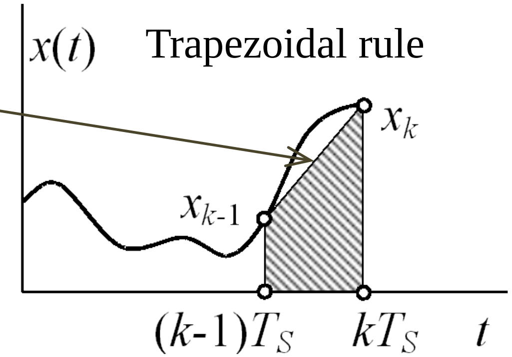
$$y_n = y_{n-1} + T_S \frac{x_{n-1} + x_n}{2}$$

Bilinear transform

$$Y(z)\left(1 - z^{-1}\right) = \frac{T_S}{2} X(z)\left(1 + z^{-1}\right) \quad \Rightarrow$$

$$\frac{Y(z)}{X(z)} = \frac{T_S}{2} \frac{1 + z^{-1}}{1 - z^{-1}} = \frac{1}{s}$$

$$\boxed{s \approx \frac{2}{T_S} \frac{1 - z^{-1}}{1 + z^{-1}}} \qquad \boxed{z \approx \frac{1 - \frac{T_S}{2} s}{1 + \frac{T_S}{2} s}}$$



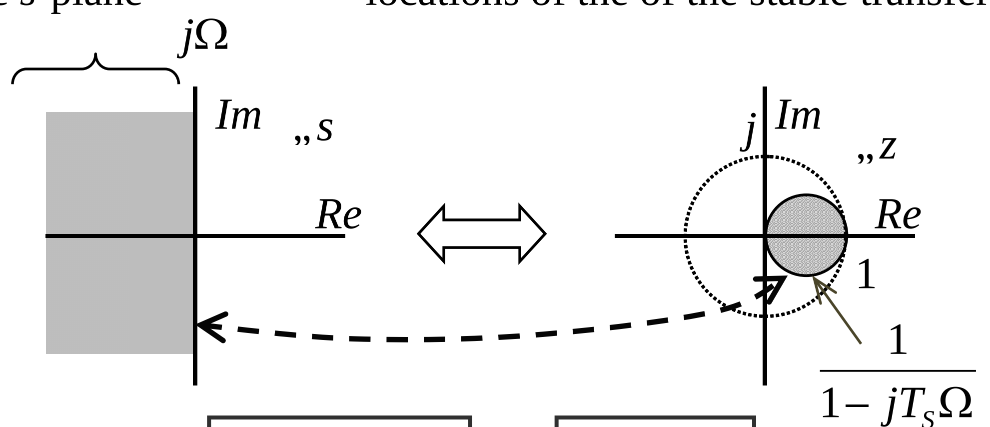
Integration approximation with the use of the rectangle rule

Mapping the s-plane imaginary axis to the z-plane for the transform based on the rectangle rule

The stable region of the *s*-plane

The unit circle as a region margin surrounding the zero locations of the of the stable transfer function denominator

Analog plane (*s*-plane)



Digital plane (*z*-plane)

$-\infty < \Omega < +\infty$

Transform based on the rectangle rule

$$\boxed{s = \frac{1}{T_S}\left(1 - z^{-1}\right)}$$

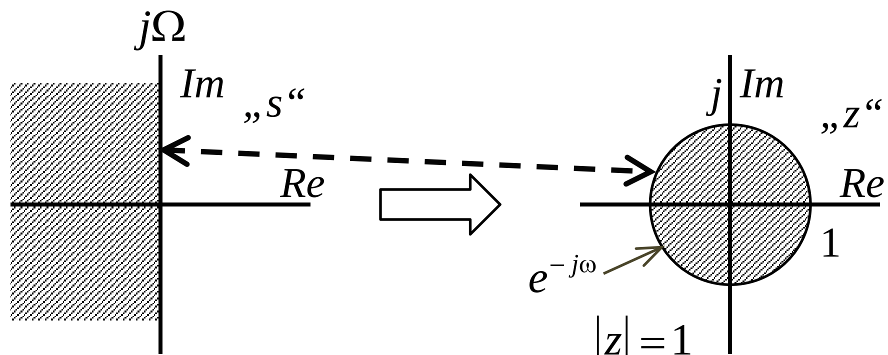
$$\boxed{z = \frac{1}{1 - T_S s}}$$

Integration approximation with the use of the trapezoidal rule

Mapping the s-plane imaginary axis to the z-plane for the transform based on the trapezoidal rule

The bilinear transform maps the s-plane to the z-plane by

$$\boxed{s=\frac{2}{T_S}\frac{1-z^{-1}}{1+z^{-1}}}$$



$$\boxed{z=\frac{1-\frac{T_S}{2}s}{1+\frac{T_S}{2}s}}$$

$$z=\left.\frac{1-j\frac{T_S}{2}\Omega}{1+j\frac{T_S}{2}\Omega}\right|_{s=j\Omega}\Rightarrow\quad |z|=\frac{\left|1-j\frac{T_S}{2}\Omega\right|}{\left|1+j\frac{T_S}{2}\Omega\right|}=1$$

Stability margin for analog and digital systems

$$z=e^{-j\omega T_S}\quad s=j\Omega\quad\Rightarrow\quad j\Omega=\frac{2}{T_S}\frac{1-e^{-j\omega T_S}}{1+e^{-j\omega T_S}}\quad\Rightarrow\quad j\Omega\frac{T_S}{2}=\frac{je^{j\omega T_S/2}\left(1-e^{-j\omega T_S}\right)/2j}{e^{j\omega T_S/2}\left(1+e^{-j\omega T_S}\right)/2}=j\,\mathrm{tg}\left(\frac{\omega T_S}{2}\right)$$

$\sin(\omega T_s/2)$

$\cos(\omega T_s/2)$

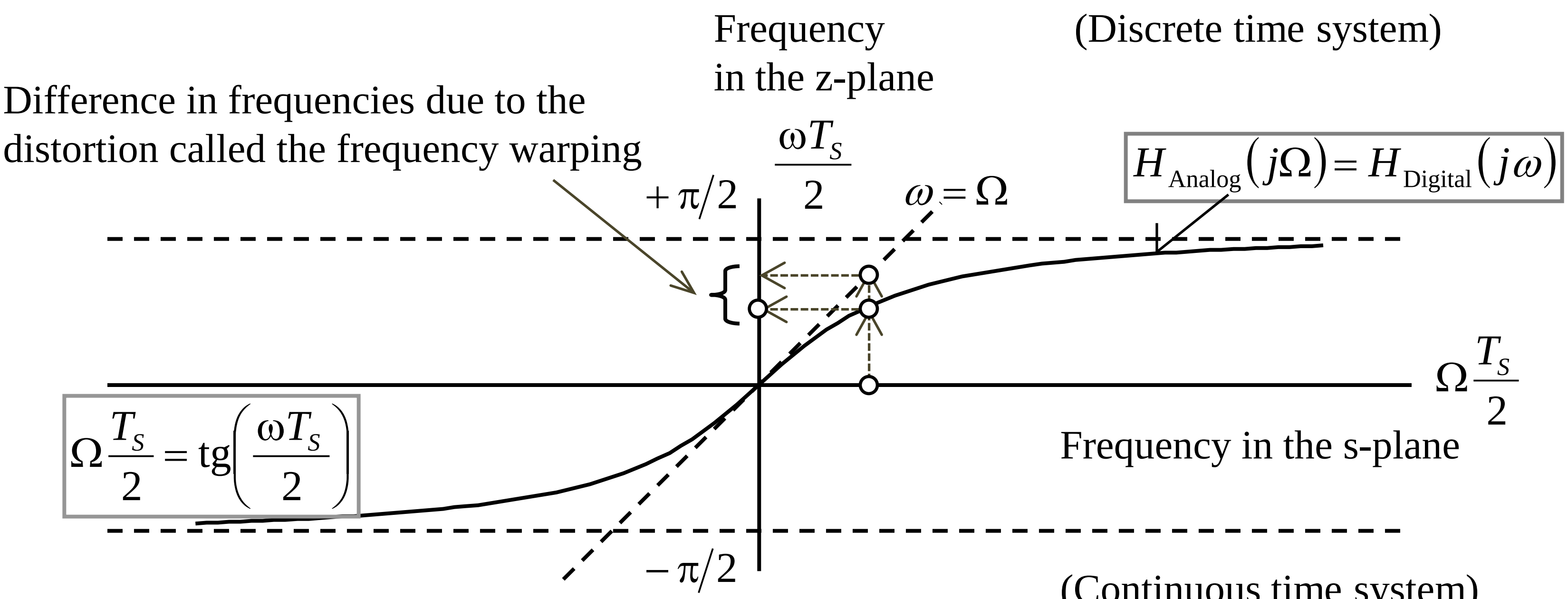
$$\Downarrow$$

$$\boxed{H_{\text{Analog}}(j\Omega)=H_{\text{Digital}}(j\omega)}$$

$$\boxed{\Omega\frac{T_S}{2}=\mathrm{tg}\left(\frac{\omega T_S}{2}\right)}$$

Frequency warping due to the bilinear transform

The bilinear transform maps the infinity interval of frequency Ω to the finite interval of frequency ω, which results in distortion



How to face to the frequency warping?

The bilinear transform maps the s-plane to the z-plane by $H(s) = H\left(\frac{2}{T_S}\frac{1-z^{-1}}{1+z^{-1}}\right)$

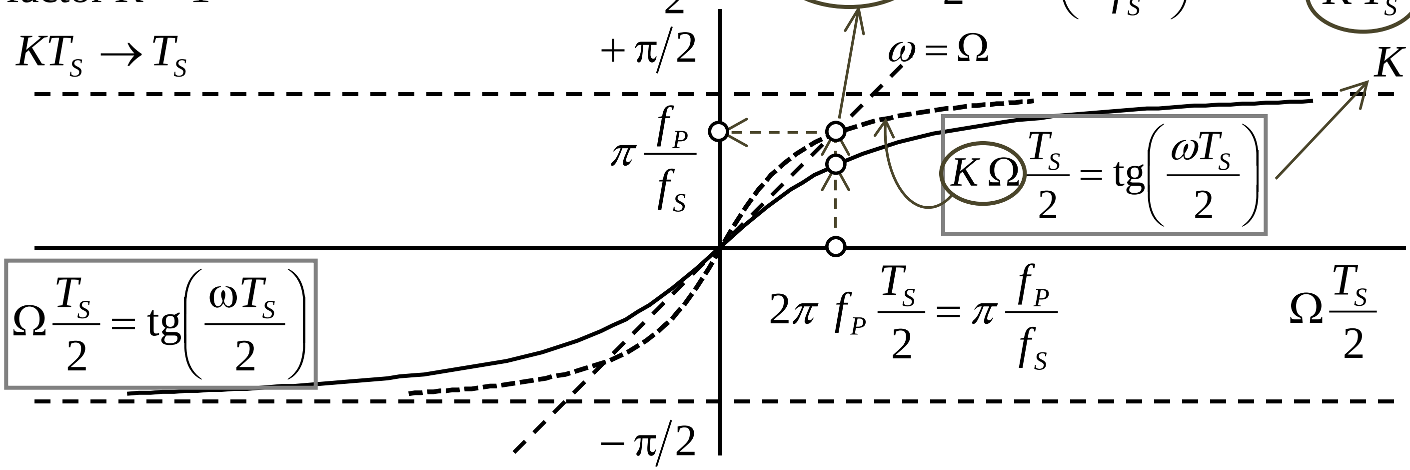
Correction factor

Introducing a correction factor $K > 1$

$KT_S \to T_S$

$$K\,2\pi f_P \frac{T_S}{2} = \mathrm{tg}\left(\frac{\pi f_P}{f_S}\right) \Rightarrow \frac{2}{K\,T_S} = 2\pi f_P \Big/ \mathrm{tg}\left(\frac{\pi f_P}{f_S}\right)$$

$K\,\Omega \to \Omega,\ K\,s \to s$



$$K\,s = \frac{2}{T_S}\frac{1-z^{-1}}{1+z^{-1}}$$

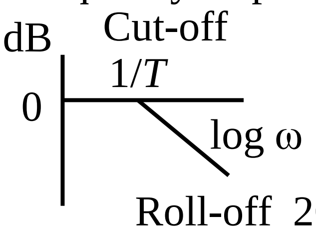
$$s = \frac{2}{K\,T_S}\frac{1-z^{-1}}{1+z^{-1}}$$

The bilinear transform can accept an parameter that specifies prewarping frequency f_p in Hz, that is a match frequency, for which the frequency responses before and after mapping match exactly.

$$H(s) = H\left(\frac{2\pi f_p}{\tan(\pi f_p / f_S)}\frac{1-z^{-1}}{1+z^{-1}}\right)$$

The first order linear filter

Frequency response



$$2\pi\, f_P = \frac{1}{T} \quad \Rightarrow \quad f_P = \frac{1}{2\pi T}$$

$$K\,T_S = 2T\,\mathrm{tg}\left(\frac{T_S}{2T}\right)$$

Transfer function of s

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{1+sT} \quad \rightarrow$$

Transfer function of z

$$H^*(z) = \frac{Y(z)}{X(z)} = \frac{1}{1+\frac{2}{K\,T_S}\frac{1-z^{-1}}{1+z^{-1}}T} = \frac{b_0 + b_1 z^{-1}}{1+a_1 z^{-1}}$$

Filter coefficients

$$b_0 = b_1 = \frac{\mathrm{tg}(T_S/2T)}{\mathrm{tg}(T_S/2T)+1}, \quad a_1 = \frac{\mathrm{tg}(T_S/2T)-1}{\mathrm{tg}(T_S/2T)+1}$$

Difference equation

$$y_n = b_0 x_n + b_1 x_{n-1} - a_1 y_{n-1}$$

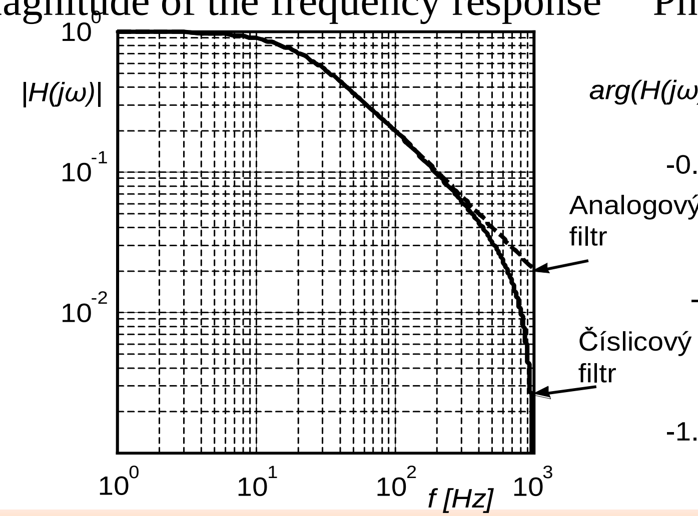
An example

The cut-off and sampling frequency is as follows

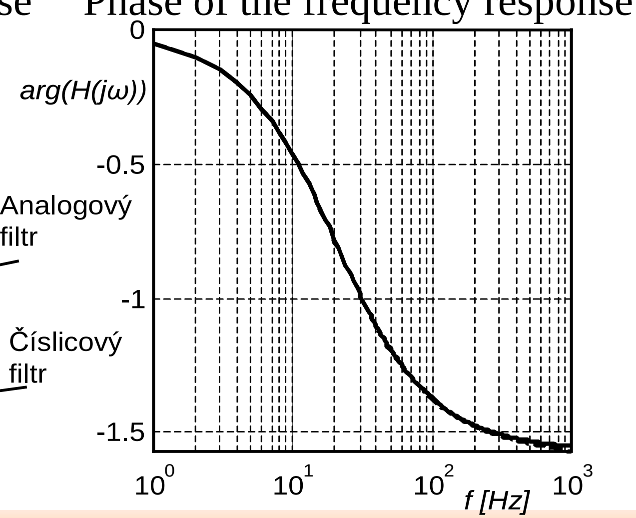
$$f_P = 20\ \mathrm{Hz}$$

$$f_S = 2000\ \mathrm{Hz}$$

Magnitude of the frequency response



Phase of the frequency response



Notch filter as a tool to filter out a spectrum component

Notch filter transfer function

$$H(s)=\frac{Y(s)}{X(s)}=\frac{1+s^2/\omega_0^2}{1+2\xi\, s/\omega_0+s^2/\omega_0^2}$$

$\boxed{H(j\omega_0)=0}$ ω_0 … the frequency to be filter out

$$\omega_0=2\pi\, f_0=2\pi\, f_P \quad\Rightarrow\quad f_P=\frac{\omega_0}{2\pi}$$

$$K\,T_S=2\,\mathrm{tg}\left(\frac{T_S}{2}\omega_0\right)\Big/\omega_0$$

The transfer function od the digital filter

$$H^*(z)=\frac{Y(z)}{X(z)}=\frac{b_0+b_1z^{-1}+b_0z^{-2}}{1+a_1z^{-1}+a_2z^{-2}}$$

$$y_n=b_0x_n+b_1x_{n-1}+b_0x_{n-2}-a_1y_{n-1}-a_2y_{n-2}$$

An example $f_P=f_0=50\text{ Hz}$

$f_S=200\text{ Hz}$

$\xi=0.05$

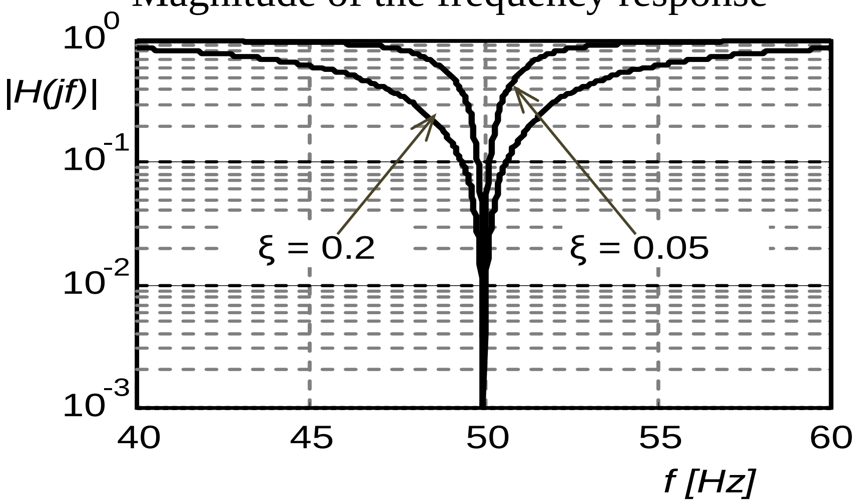
FIR filter coefficients

$$b_0=\frac{(KT_S)^2+4/\omega_0^2}{(KT_S)^2+4\xi\,KT_S/\omega_0+4/\omega_0^2}$$

$$a_1=b_1=2\frac{(KT_S)^2-4/\omega_0^2}{(KT_S)^2+4\xi\,KT_S/\omega_0+4/\omega_0^2}$$

$$a_2=\frac{(KT_S)^2-4\xi\,KT_S/\omega_0+4/\omega_0^2}{(KT_S)^2+4\xi\,KT_S/\omega_0+4/\omega_0^2}$$

Magnitude of the frequency response



Filter of the moving average type

A moving average of M samples corresponds to the difference equation

$$y_n = \frac{1}{M}\left(x_n + x_{n-1} + \ldots + x_{n-M+1}\right)$$

Z-transform

$$H(z) = \frac{1}{M}\left(1 + z^{-1} + \ldots + z^{-(M-1)}\right) = \frac{1}{M}\frac{z^{M-1} + z^{M-2} + \ldots + 1}{z^{M-1}}$$

The sum of the finite numbers in a geometric progression

$$H(z) = \frac{1}{M}\frac{1 - z^{-M}}{1 - z^{-1}}$$

Frequency transfer function

$$H(j\omega) = \frac{1}{M}\frac{1 - \exp\left(-j\omega T_S M\right)}{\left(1 - \exp\left(-j\omega T_S\right)\right)}$$

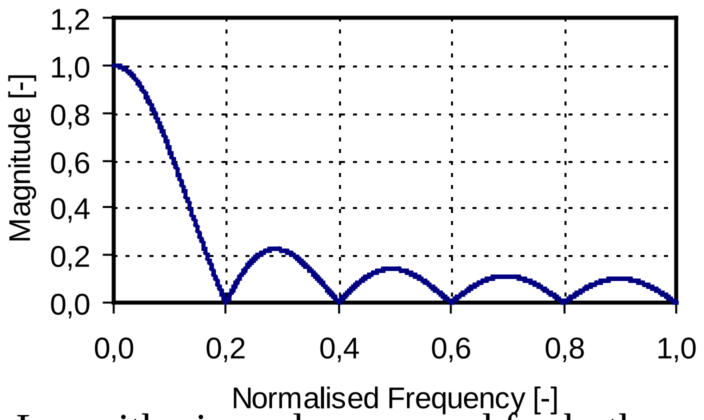
Note that it can be shown

$$H(z)\big|_{z=1} = H(j\omega)\big|_{\omega=0} = 1 \qquad \text{LP filter}$$

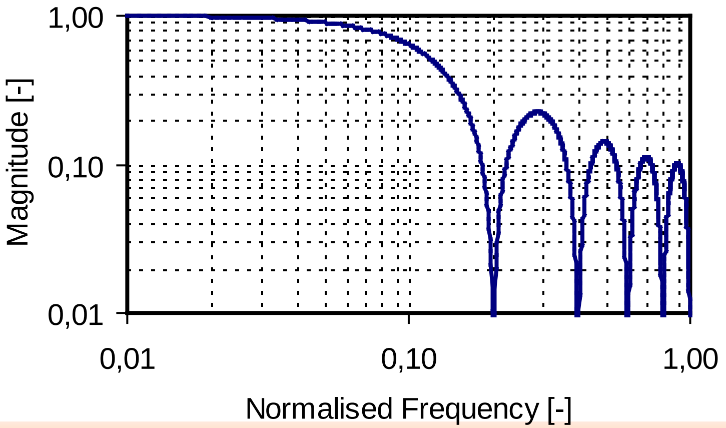
LP filter zeros

$$H(j2\pi f) = 0 \quad \Rightarrow \quad f/f_S = k/M\,,\ k = 1, 2, \ldots, M$$

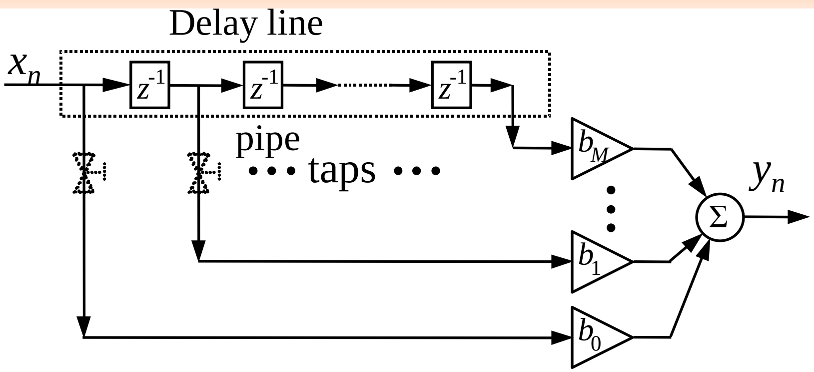
Frequency response function of the normalized frequency (f/f_S)



Logarithmic scales are used for both the frequency- and magnitude- axes



FIR filters



Difference equation ($M+1$ taps on a pipe)

$$y_n = b_0 x_n + b_1 x_{n-1} + \ldots + b_M x_{n-M}$$

Z-transform

$$H(z) = b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}$$

Frequency transfer function

$$H(j\omega) = b_0 + b_1 \exp(-j\omega T_S) + \ldots$$
$$+ b_M \exp(-j\omega T_S M)$$

After rewriting ……

$$H(j\omega) = \exp(-j\omega T_S M/2)$$
$$\left[b_0 \exp(j\omega T_S M/2) + b_M \exp(-j\omega T_S M/2) + \right.$$
$$\left. + b_1 \exp(j\omega T_S (M/2-1)) + b_{M-1} \exp(-j\omega T_S (M/2-1)) + \ldots . \right]$$

Euler's formulas

$$\cos(\alpha) + j\sin(\alpha) = \exp(j\alpha)$$
$$\cos(\alpha) - j\sin(\alpha) = \exp(-j\alpha)$$
$$\sin(\alpha) = (\exp(j\alpha) - \exp(-j\alpha))/2j$$
$$\cos(\alpha) = (\exp(j\alpha) + \exp(-j\alpha))/2$$

Coefficient even symmetry $b_0 = b_M,\ b_1 = b_{M-1}, \ldots$

$$H(j\omega) = \exp(-j\omega T_S M/2)$$
$$\left[b_0 \cos(\omega T_S M/2)/2 + b_1 \cos(\omega T_S (M/2-1))/2 + \ldots\right]$$

Coefficient odd symmetry $b_0 = -b_M,\ b_1 = -b_{M-1}, \ldots$

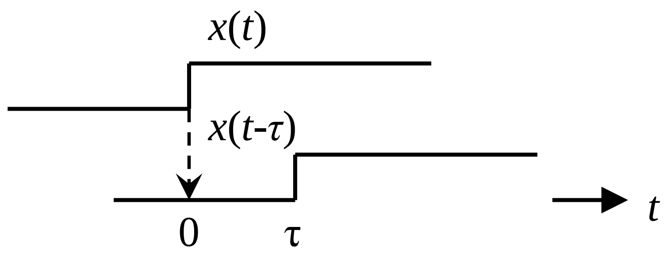
$$H(j\omega) = \exp(-j(\omega T_S M/2 + \pi/2))$$
$$\left[b_0 \sin(\omega T_S M/2)/2 + b_1 \sin(\omega T_S (M/2-1))/2 + \ldots\right]$$

It can be concluded that the FIR filter phase is a linear function of frequency ω

$$\boxed{\varphi(\omega) = \begin{cases} -\omega T_S M/2, & \text{even symmetry} \\ -\omega T_S M/2 - \pi/2, & \text{odd symmetry.} \end{cases}}$$

Properties of linear phase filters

An example of the linear phase filter or system is a delay



The time delay transfer function $H(\omega) = \exp(-j\omega T_D)$ (only delay, no distortion of the signal)

The phase of the delay transfer function $\varphi(\omega) = -\omega T_D$ is a linear function of ω

The time delay does not distort signal in the pass band frequency

> Signals are not distorted by filtration if the phase is a linear function of frequency

Group delay $\tau_g(\omega) = -\dfrac{d\varphi(\omega)}{d\omega}$

Phase delay $\tau_\Phi(\omega) = -\dfrac{\varphi(\omega)}{\omega}$

Time delay: $\tau_g(\omega) = T_D$

$\tau_\Phi(\omega) = T_D$

FIR filter summary

The FIR filter is described by the difference equation

$$y_n = b_0 x_n + b_1 x_{n-1} + \ldots + b_{M-1} x_{n-M+1} + b_M x_{n-M}$$

Any input sequence, containing the samples of the limited value, produces the output sequence, which is limited as well. The FIR filter is a stable system.

Properties of the FIR filter coefficients $b_0, b_1, \ldots, b_{M-1}, b_M$

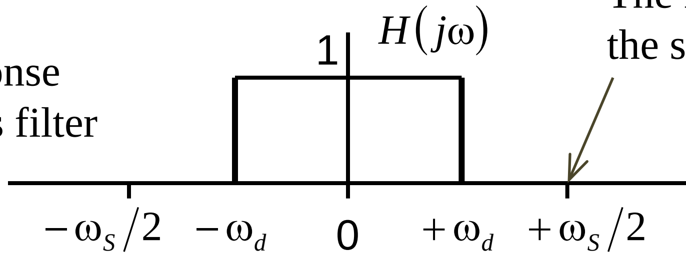
<table>
<tr><th>Filter Type</th><th>Order M</th><th>Coefficient symmetry</th><th>$H(0)$</th><th>$H(f_S/2)$</th></tr>
<tr><td>Type I</td><td>even</td><td rowspan="2">even
$b_0 = b_M,\ b_1 = b_{M-1}, \ldots$</td><td rowspan="2">arbitrary</td><td>arbitrary</td></tr>
<tr><td>Type II</td><td>odd</td><td rowspan="2">$H(f_S/2) = 0$</td></tr>
<tr><td>Type III</td><td>even</td><td rowspan="2">Odd (anti-symmetry)
$b_0 = -b_M,\ b_1 = -b_{M-1}, \ldots$</td><td rowspan="2">$H(0) = 0$</td></tr>
<tr><td>Type IV</td><td>odd</td><td>arbitrary</td></tr>
</table>

Ideal low pass filter

The ideal low pass filter can be designed in the frequency domain. The impulse response is obtained by the inverse Fourier transform.

The Nyquist frequency is equal to half the sampling frequency

The frequency response of the ideal low pass filter



$$H(j\omega)=\begin{cases}1, & -\omega_d \le \omega \le \omega_d \\ 0, & \omega_S/2 > \omega > \omega_d,\ -\omega_S/2 < \omega < -\omega_d\,.\end{cases}$$

The inverse Fourier transform of the frequency response results in

$$h_n=\frac{1}{2\pi}\int_{-\frac{\omega_S}{2}}^{\frac{\omega_S}{2}} H(j\omega)\exp(j\omega nT_S)\,\mathrm{d}\,\omega=\int_{-\frac{f_S}{2}}^{\frac{f_S}{2}} H(j2\pi f)\exp(j2\pi f\,nT_S)\,\mathrm{d}\,f=\int_{-f_d}^{f_d}\exp(j2\pi fnT_S)\,\mathrm{d}\,f=$$

$$=\frac{\sin(2\pi n f_d T_S)}{\pi n},\qquad n=0,\pm1,\pm2,\ldots$$

The ideal low pass filter is a non-causal system of the infinity order.

FIR low-pass filter

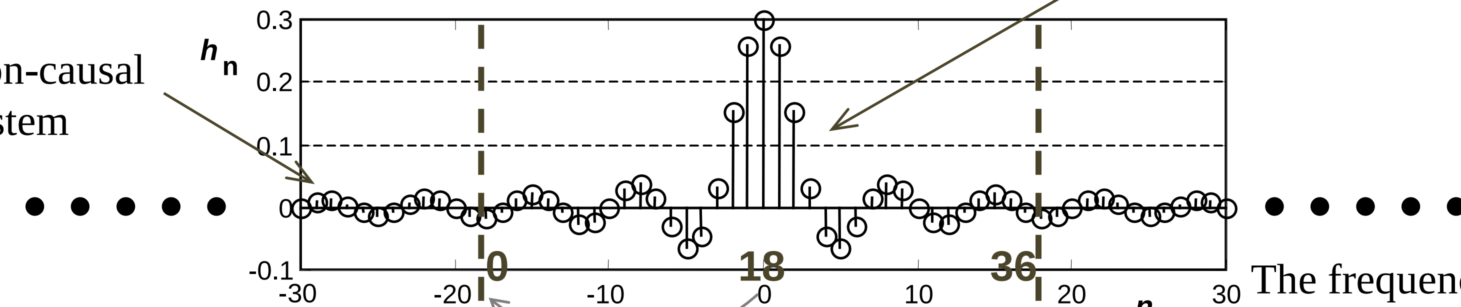
The ideal low pass filter of the corrected order, which is restricted to the finite number, can be used as a sufficiently good low pass filter.

The impulse response of the ideal low pass filter

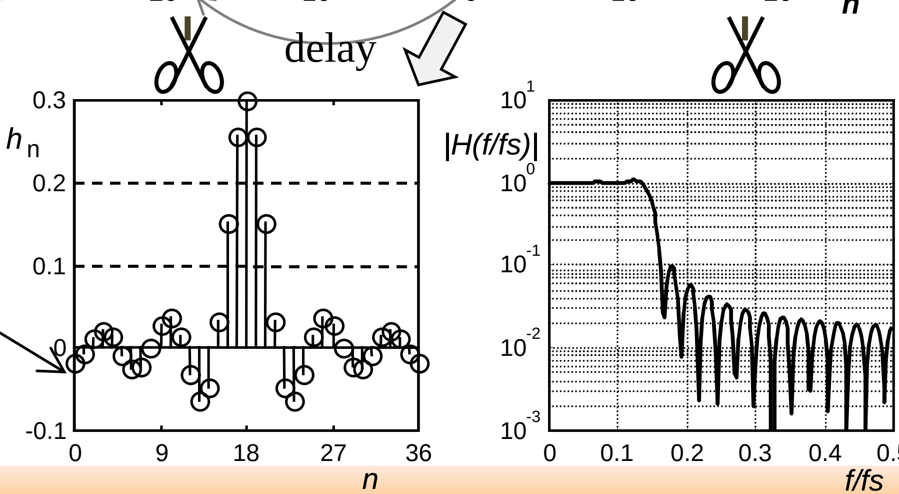
$$h_n = \frac{\sin(2\pi n f_d T_S)}{\pi n}, \quad n = 0, \pm 1, \pm 2, \ldots$$

Symmetry of coefficients is determining the linear phase of the filter

Non-causal system



The frequency response of the low-pass filter



Delay transforms non-causal system to a causal system

Non-Causal FIR filter

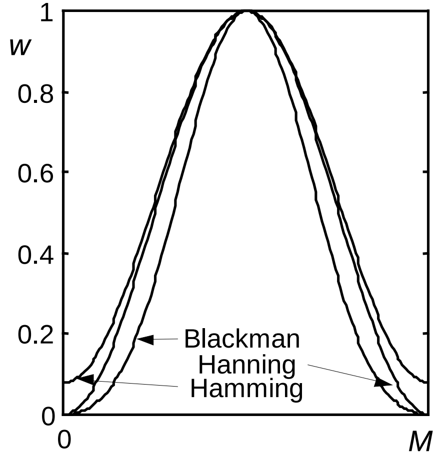
$$y_n = \sum_{i=-M/2}^{+M/2} h_i x_{n-i}$$

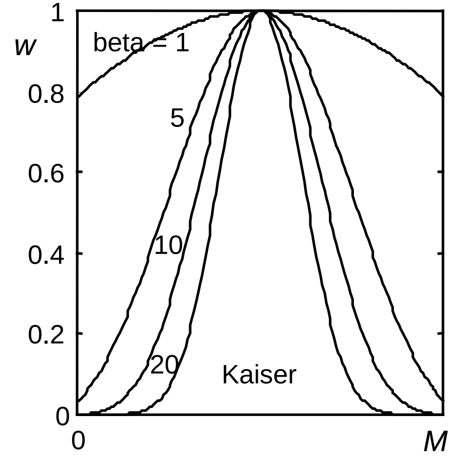
Causal FIR filter

$$y_n = \sum_{i=0}^{M} h_i x_{n-i}$$

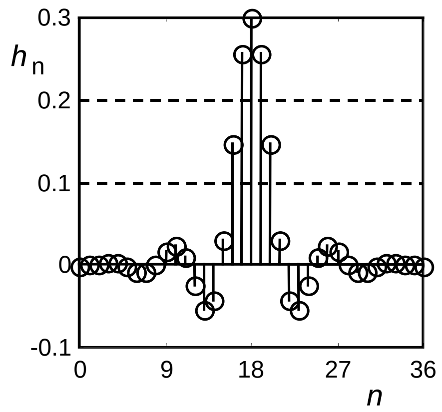
Time windows for the filter impulse response

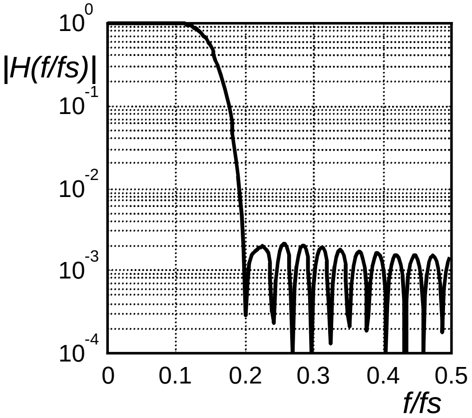
Windowing of the impulse responses improves the frequency response of the FIR filters





The low pass filter design using the Haning window





Frequency transform

High pass, band-pass and band-stop filters are derived from the low pass filter by using frequency transformation

Mapping $\hat{s}$ onto s .... $s = F(\hat{s})$

The transfer functions are related through

$$H_D(\hat{s}) = H_{LP}(s)\Big|_{s=F(\hat{s})}$$

$$H_{LP}(s) = H_{LP}(\hat{s})\Big|_{\hat{s}=F^{-1}(s)}$$

Example:
Transform of LP to HP filter

$$H_{LP}(s) = \frac{1}{1+s}\bigg|_{s=1/\hat{s}} \Rightarrow H_{LP}\left(\frac{1}{\hat{s}}\right) = H_{HP}(\hat{s}) = \frac{1}{1+1/\hat{s}} = \frac{\hat{s}}{1+\hat{s}}$$

$$H_{LP}(j\Omega) = \frac{1}{1+j\Omega}\bigg|_{j\Omega=-j/\hat{\Omega}} \Rightarrow H_{LP}\left(\frac{-j}{\hat{\Omega}}\right) = H_{HP}(j\hat{\Omega}) = \frac{1}{1+1/j\hat{\Omega}} = \frac{j\hat{\Omega}}{1+j\hat{\Omega}}$$

Highpass filter	$s = \frac{\Omega_P \hat{\Omega}_P}{\hat{s}}$	$\Omega = -\frac{\Omega_P \hat{\Omega}_P}{\hat{\Omega}}$	$0 \le \Omega \le \Omega_P \Rightarrow -\infty \le \hat{\Omega} \le -\hat{\Omega}_P$ $-\Omega_P \le \Omega \le 0 \Rightarrow \hat{\Omega}_P \le \hat{\Omega} \le \infty$
Bandpass filter	$s = \Omega_0 \frac{\hat{s}^2 + \hat{\Omega}_0^2}{\hat{s}(\hat{\Omega}_{P2} - \hat{\Omega}_{P1})}$	$\Omega = -\Omega_P \frac{\hat{\Omega}_0^2 - \hat{\Omega}^2}{\hat{\Omega}(\hat{\Omega}_{P2} - \hat{\Omega}_{P1})}$	$\hat{\Omega}_{P1}\hat{\Omega}_{P2} = \hat{\Omega}_{S1}\hat{\Omega}_{S2} = \hat{\Omega}_0^2$
Bandstop filter	$s = \Omega_0 \frac{\hat{s}(\hat{\Omega}_{S2} - \hat{\Omega}_{S1})}{\hat{s}^2 + \hat{\Omega}_0^2}$	$\Omega = \Omega_0 \frac{\hat{\Omega}(\hat{\Omega}_{S2} - \hat{\Omega}_{S1})}{\hat{\Omega}_0^2 - \hat{\Omega}^2}$	$\hat{\Omega}_{P1}\hat{\Omega}_{P2} = \hat{\Omega}_{S1}\hat{\Omega}_{S2} = \hat{\Omega}_0^2$

Design of IIR digital filters

The following steps are suppressing the frequency distortion $\Omega \frac{T_S}{2} = \mathrm{tg}\left(\frac{\omega T_S}{2}\right)$

Step 1: Prewarp the specified digital frequency specification of the desired digital filter $G_D(z)$ to arrive at the frequency specifications of an analog filter $H_D(s)$ of the same type

Step 2: Convert the frequency specifications of $H_D(s)$ into that of a prototype analog lowpass filter $H_{LP}(s)$ using an appropriate frequency transformation

Step 3: Design the analog lowpass filter $H_{LP}(s)$

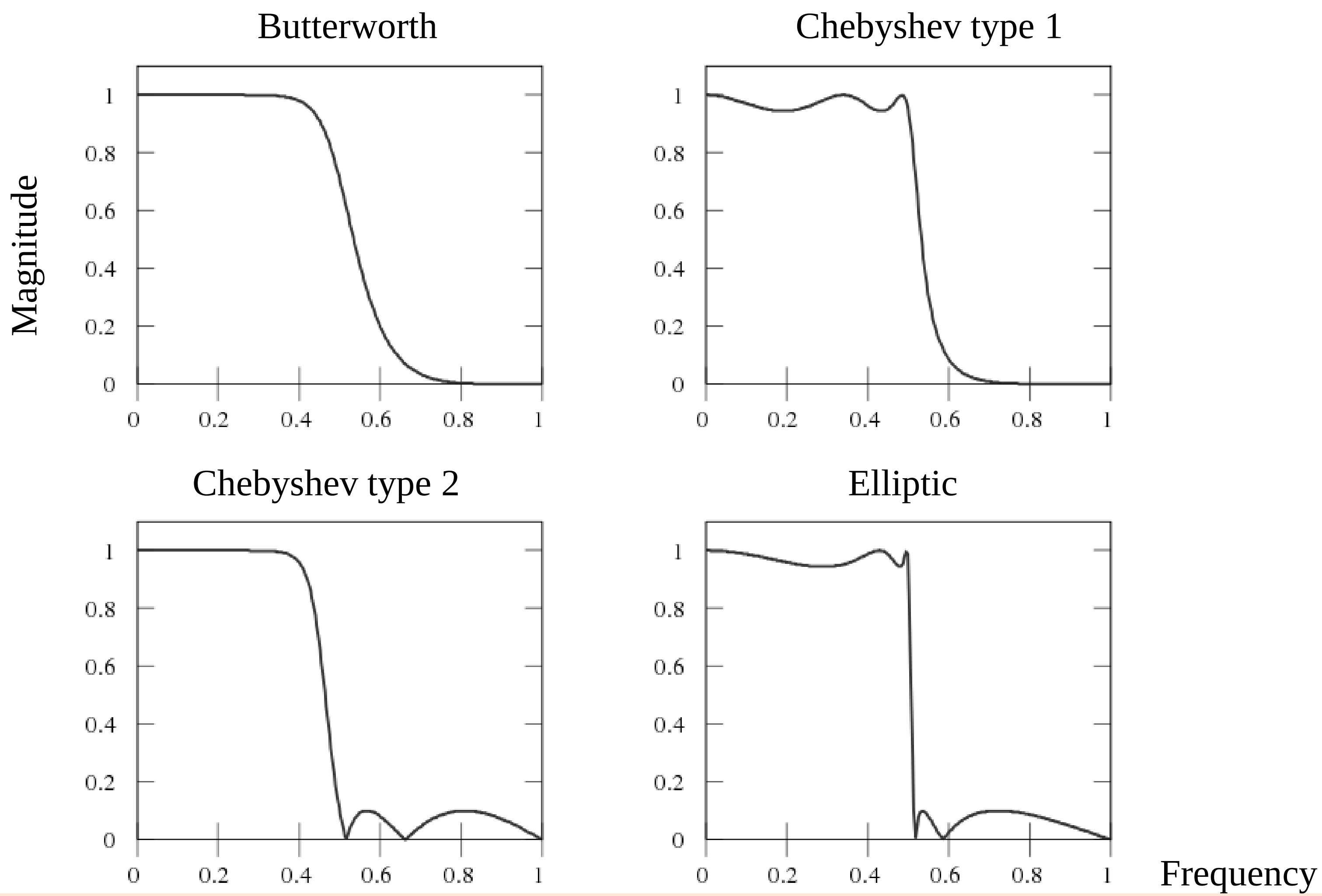
Step 4: Convert the transfer function $H_{LP}(s)$ into $H_D(s)$ using the inverse of the frequency transformation used in Step 2

Step 5: Transform the transfer function $H_D(s)$ using the bilinear transformation to arrive at the desired digital IIR transfer function $G_D(z)$

$$s = \frac{2}{T_S}\frac{1-z^{-1}}{1+z^{-1}}$$

See [Mitra]

Analog low pass linear filters



Butterworth filters

The gain $G(\omega)$ of the n-th order Butterworth low pass filter is given in terms of the transfer function $H(s)$ as:

$$G^2(\omega) = |H(\omega)|^2 = \frac{G_0^2}{1 + (\omega/\omega_C)^{2n}}$$

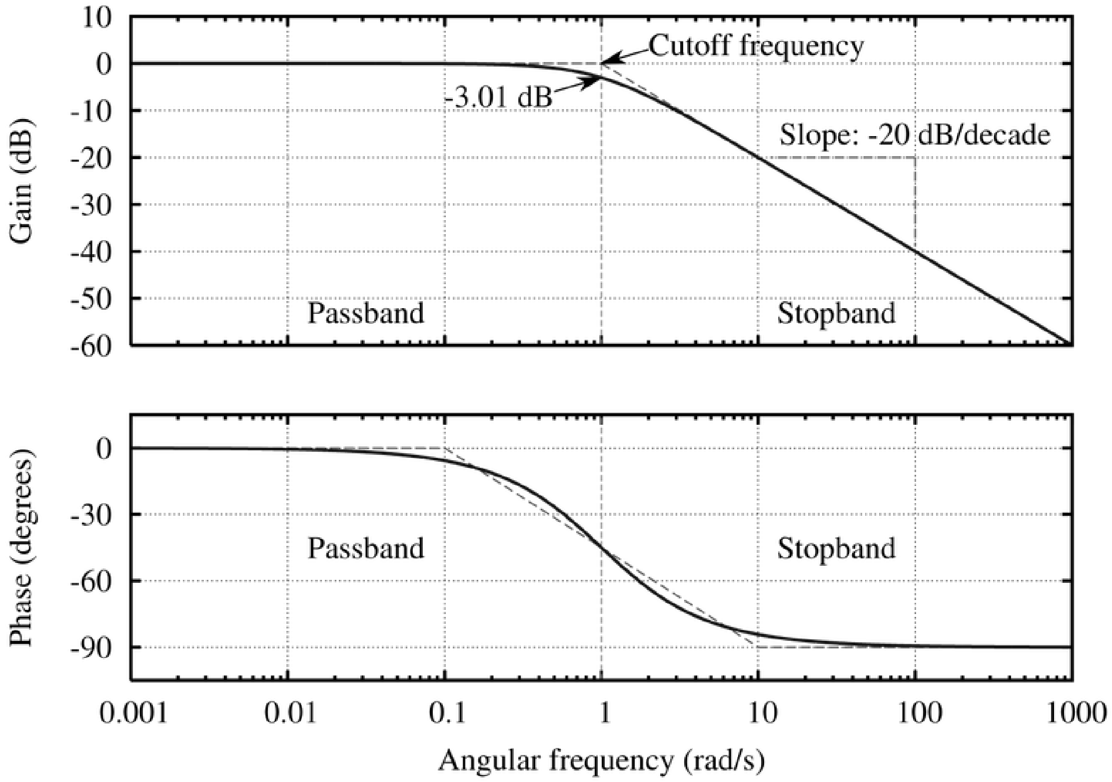
where

n ... order of filter

ω_c ... cutoff frequency (approximately the -3dB frequency)

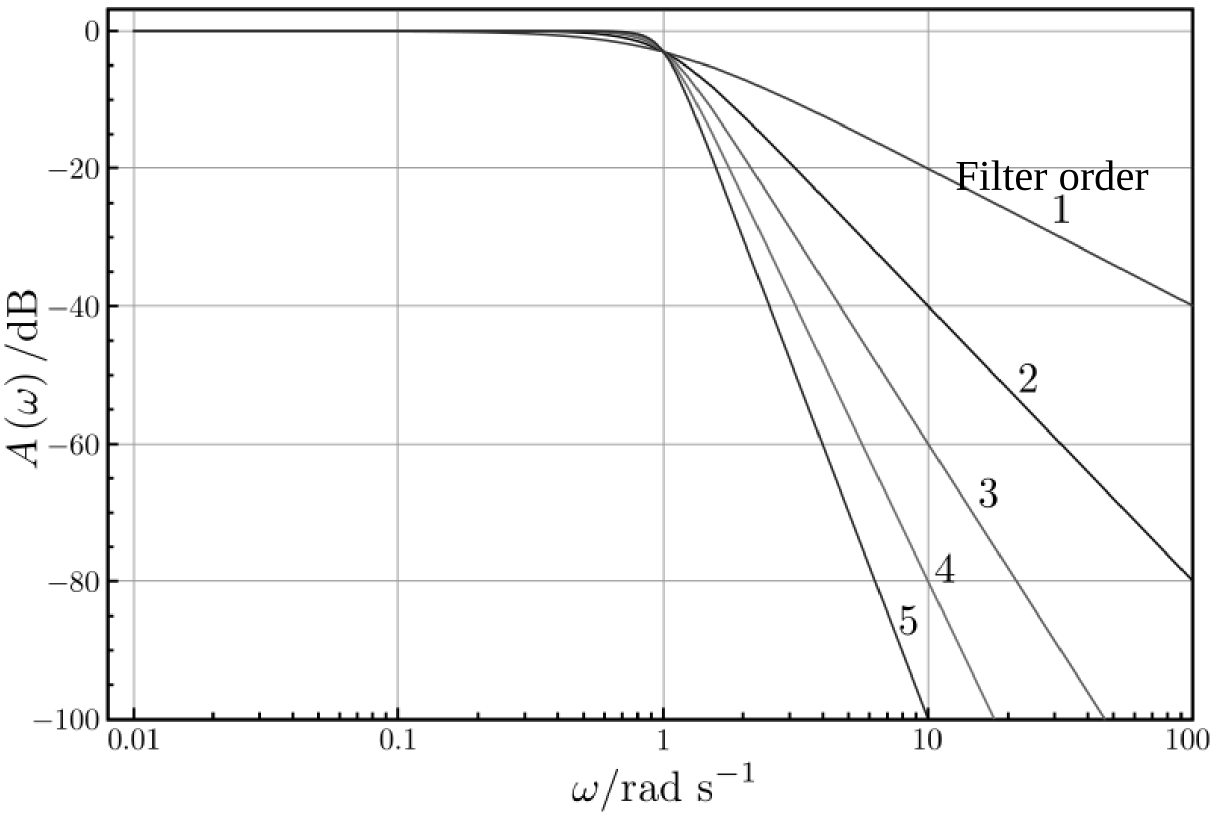
G_0 ... the DC gain (gain at zero frequency)

The first order filter



See [http://en.wikipedia.org/wiki/Butterworth_filter]

High order Butterworth filters



See [http://en.wikipedia.org/wiki/Butterworth_filter]

Butterworth polynomials

We wish to determine the transfer function $H(s)$ where $s = \sigma + j\omega$. Since $H(s)H(-s)$ evaluated at $s = j\omega$ is simply equal to $|H(j\omega)|^2$, it follows that:

$$H(s)H(-s) = \frac{G_0^2}{1+\left(-s^2/\omega_C^2\right)^n}$$

The k-th pole is specified by:

$$-s^2/\omega_C^2 = (-1)^{\frac{1}{n}} = \exp\left(\frac{j(2k-1)\pi}{n}\right),\ k = 1,2,3,\ldots,n \Rightarrow \quad s_k = \omega_C \exp\left(\frac{j(2k+n-1)\pi}{2n}\right),\ k = 1,2,3,\ldots,n$$

The transfer function may be written in terms of these poles as:

$$H(s) = \frac{G_0}{\prod_{k=1}^{n}(s-s_k)/\omega_C}$$

The denominator is a Butterworth polynomial in s.

Normalized Butterworth polynomials

$\omega_C = 1$

$$B_n(s) = \begin{cases} \prod_{k=1}^{n/2}\left(s^2 - 2s\cos\left(\frac{2k+n-1}{2n}\pi\right)+1\right) & \text{for } n \text{ even} \\ (s+1)\prod_{k=1}^{(n-1)/2}\left(s^2 - 2s\cos\left(\frac{2k+n-1}{2n}\pi\right)+1\right) & \text{for } n \text{ odd} \end{cases}$$

See [http://en.wikipedia.org/wiki/Butterworth_filter]

Normalized Butterworth polynomials

$\omega_C = 1$

n	Factors of Polynomial $B_n(s)$
1	$(s + 1)$
2	$s^2 + 1.4142s + 1$
3	$(s + 1)(s^2 + s + 1)$
4	$(s^2 + 0.7654s + 1)(s^2 + 1.8478s + 1)$
5	$(s + 1)(s^2 + 0.6180s + 1)(s^2 + 1.6180s + 1)$
6	$(s^2 + 0.5176s + 1)(s^2 + 1.4142s + 1)(s^2 + 1.9319s + 1)$
7	$(s + 1)(s^2 + 0.4450s + 1)(s^2 + 1.2470s + 1)(s^2 + 1.8019s + 1)$
8	$(s^2 + 0.3902s + 1)(s^2 + 1.1111s + 1)(s^2 + 1.6629s + 1)(s^2 + 1.9616s + 1)$

See [http://en.wikipedia.org/wiki/Butterworth_filter]

Chebyshev filters of the type 1

The gain (or amplitude) response as a function of angular frequency ω of the *n*-th order low pass filter is

$$G_n^2(\omega) = |H_n(\omega)|^2 = \frac{1}{1+\varepsilon^2 T_n^2(\omega/\omega_0)}$$

where ε is the ripple factor, ω_0 is the cutoff frequency and $T_n(\ldots)$ is the Chebyshev polynomial of the *n*-th order.

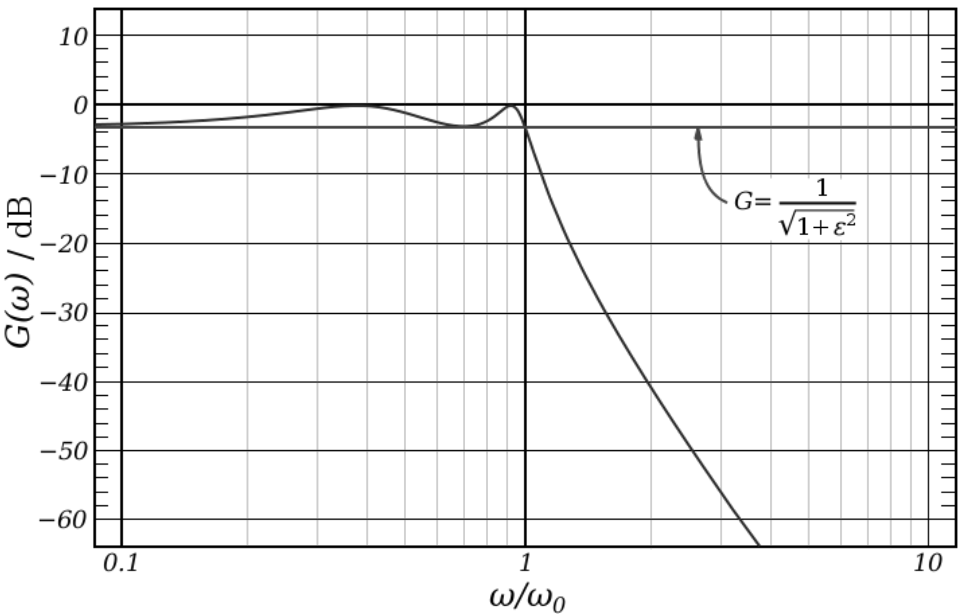
$$T_0(x) = 1$$
$$T_1(x) = x$$
$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

Trigonometric definition

$$T_n(x) = \begin{cases} \cos(n\cos^{-1}(x)), & |x| \le 1 \\ \cosh(n\cosh^{-1}(x)), & |x| > 1 \end{cases}$$

$$T_n(\cos(\upsilon)) = \cos(n\upsilon)$$

Ripple in dB $\quad 20\log_{10}\dfrac{1}{\sqrt{1+\varepsilon^2}}$



See [http://en.wikipedia.org/wiki/Chebyshev_filter]

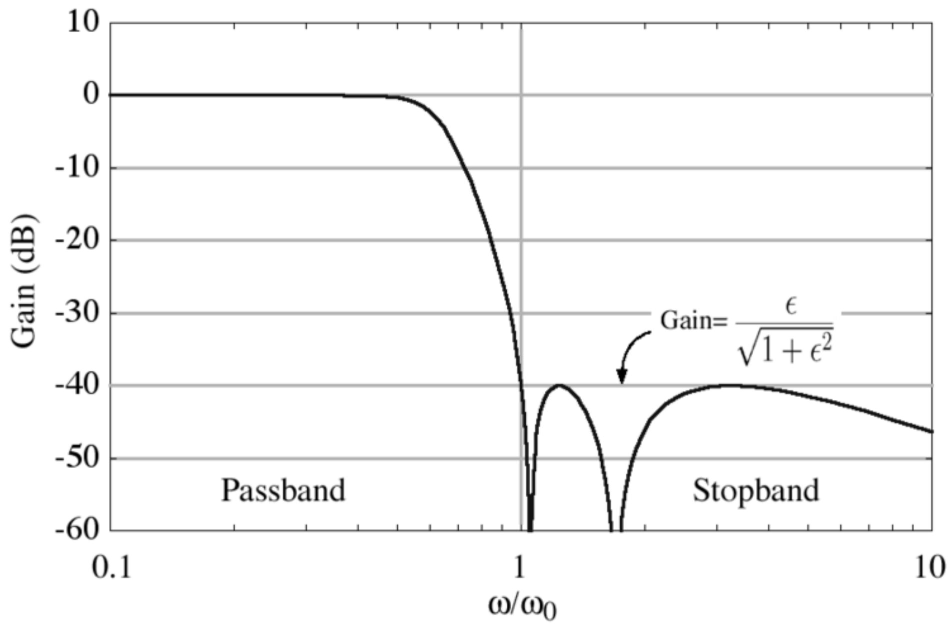
Chebyshev filters of the type 2

The gain (or amplitude) response as a function of angular frequency ω of the *n*-th order low pass filter is

$$G_n^2(\omega) = \left|H_n(\omega)\right|^2 = \frac{1}{1 + 1/\left(\varepsilon^2 T_n^2(\omega/\omega_0)\right)}$$

where ε is the ripple factor, ω_0 is the cutoff frequency and $T_n(...)$ is a Chebyshev polynomial of the *n*-th order.

Stop band ripple $20\log_{10}\frac{\varepsilon}{\sqrt{1+\varepsilon^2}}$



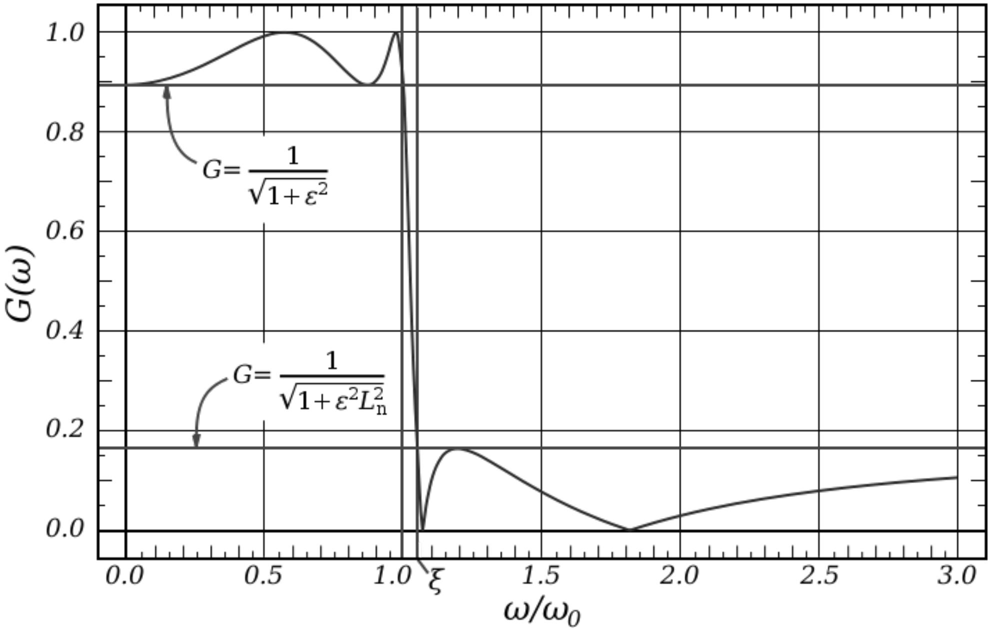
See [http://en.wikipedia.org/wiki/Chebyshev_filter]

Elliptic filters

The gain (or amplitude) response as a function of angular frequency ω of the n-th order low pass filter is

$$G_n^2(\omega) = \left|H_n(\omega)\right|^2 = \frac{1}{1+\varepsilon^2 R_n^2(\xi, \omega/\omega_0)}$$

where ε is the ripple factor, ξ is the selectivity factor , ω_0 is the cutoff frequency and R_n is the n-th order elliptic rational function of angular frequency ω.



See [http://en.wikipedia.org/wiki/Elliptic_filter]

Comparison of FIR and IIR filters

	Advantages	Disadvantages
FIR filters	Always stable Linear phase Possible to design any frequency response	High order Large number of coefficients
IIR filters	Possible instability Small number of coefficients	Low order Non-linear phase Overflow of an accumulator

Digital signal processors (DSP)

ADSP 2185 type (fixed point math), special unit MAC for computation of a formula

$$y_n = \sum_{i=0}^{taps-1} b_i x_{n-i}$$

MAC statements

```
[IF cond] MR    = X*Y; or = X*X;              (SS) or (SU) or
or |AF|         = MR+X*Y; or = MR+X*X;        (US) or (UU) or
                = MR- X*Y; or = MR- X*X;      (RND)
                = MR[(RND)];
                =0;
IF MV SAT MR;
```

En example of the code for a FIR filter

```
      ......
      I4=^fir_coefs;  M4=1;   L4=taps;
      CNTR=taps-1;
      MR=0, MX0=DM(I0,M0), MY0=PM(I4,M4);
      DO firloop UNTIL CE;
firloop: MR=MR+MX0*MY0(SS), MX0=DM(I0,M0), MY0=PM(I4,M4);
      { IF NOT CE JUMP firloop;}
      MR=MR+MX0*MY0(RND);
      IF MV SAT MR;
      ......
```

DSP Hardware tools for filters

- Cyclic buffers
- Indirect addressing using pointers
- Index registers with automatic indirect address increments

Allpass filters

The transfer function $G(z)$ of a IIR filter is called an allpass transfer function if the magnitude of the frequency response is equal to unit for all frequencies

$$\left|G\left(e^{j\omega T_S}\right)\right| = 1, \quad \text{for all } \omega$$

An M-th order causal real-coefficient allpass transfer function is of the form

$$G_M(z) = \frac{d_M + d_{M-1}z^{-1} + \ldots + d_1 z^{-M+1} + z^{-M}}{1 + d_1 z^{-1} + \ldots + d_{M-1} z^{-M+1} + d_M z^{-M}}$$

If we denote the denominator as

$$D_M(z) = 1 + d_1 z^{-1} + \ldots + d_{M-1} z^{-M+1} + d_M z^{-M}$$

Then $G_M(z)$ can be written

$$G_M(z) = \frac{z^{-M} D_M\left(z^{-1}\right)}{D_M(z)}$$

If $z = r \exp(j\varphi)$ is a pole of a real-coefficient allpass transfer function then it has a zero $z = 1/r \exp(-j\varphi)$

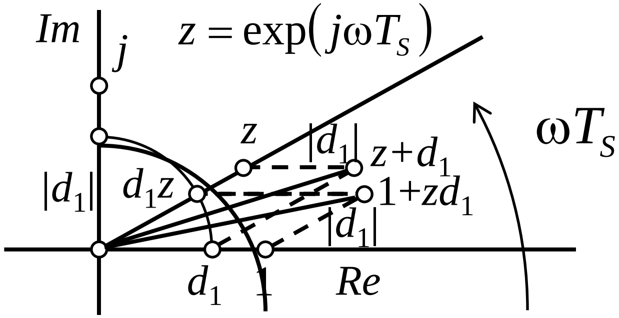
The first order allpass filter

$$G_1(z) = \frac{d_1 + z^{-1}}{1 + d_1 z^{-1}} \qquad G_1(z) = \frac{z d_1 + 1}{z + d_1}$$

See [Mitra]

Digital plane „z“

Unit circle



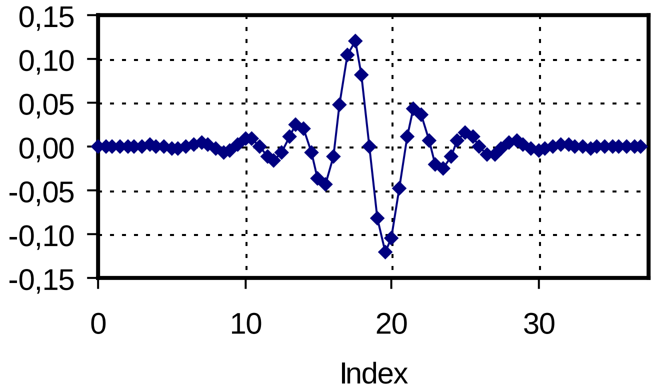
Congruency of triangles shows that $|zd_1 + 1| = |z + d_1|$

$$\left|G_1\left(e^{j\omega T_S}\right)\right| = 1$$

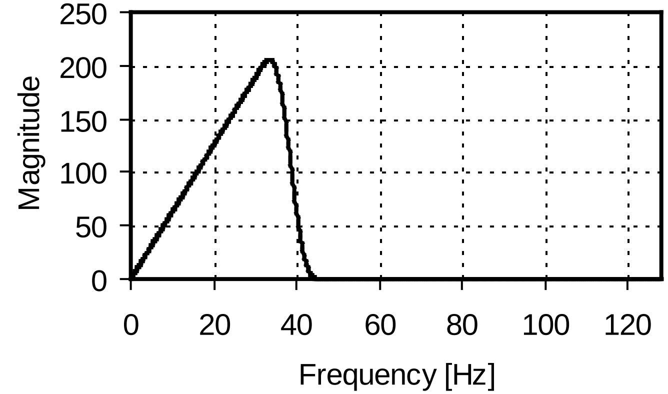
FIR filter as a differentiator

FIR filter differentiator (Kaiser window)

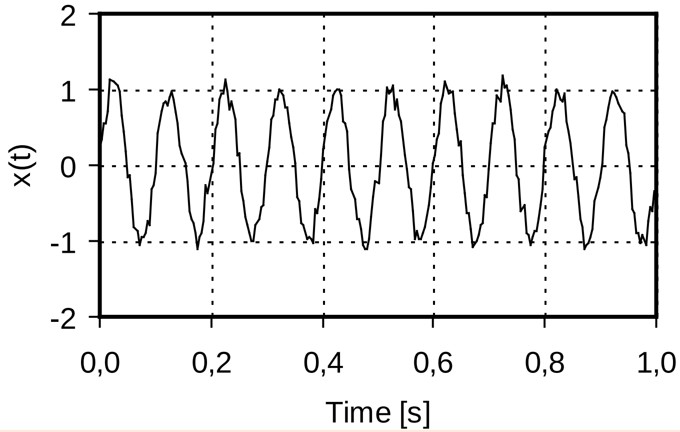
Impulse response



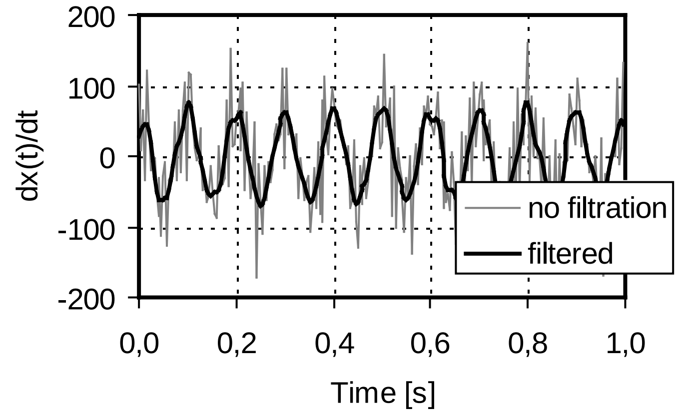
Frequency response



Input signal

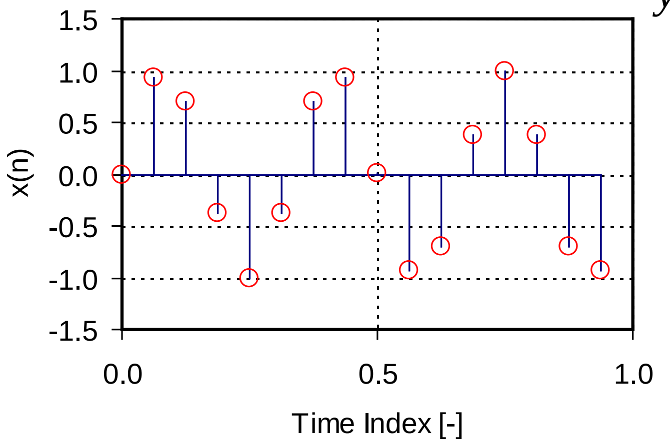


The first derivative with respect to time

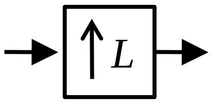


Up-sampler & down-sampler

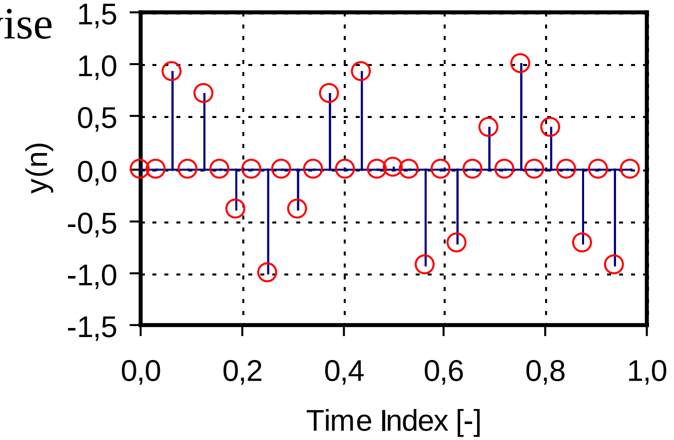
$$y_n = \begin{cases} x_{n/L}, & n = 0, \pm L, \pm 2L, \ldots \\ 0, & \text{otherwise} \end{cases}$$

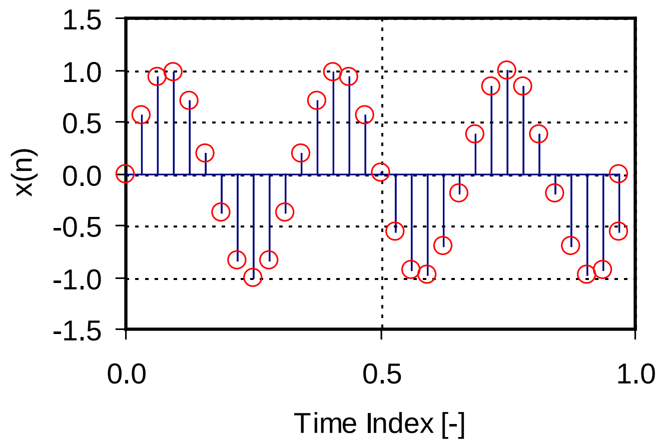


An up-sampler



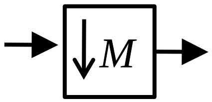
$L = 2$



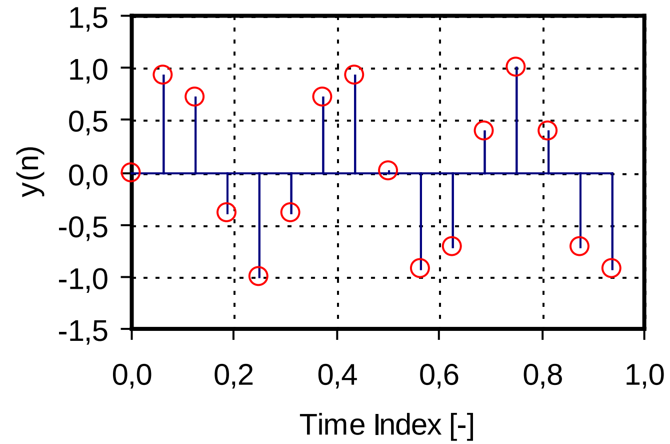


$$y_n = x_{nM}$$

An down-sampler



$M = 2$



En example of multirate filters

A standard for vibration testing of rolling-bearings requires the pass-band filter for

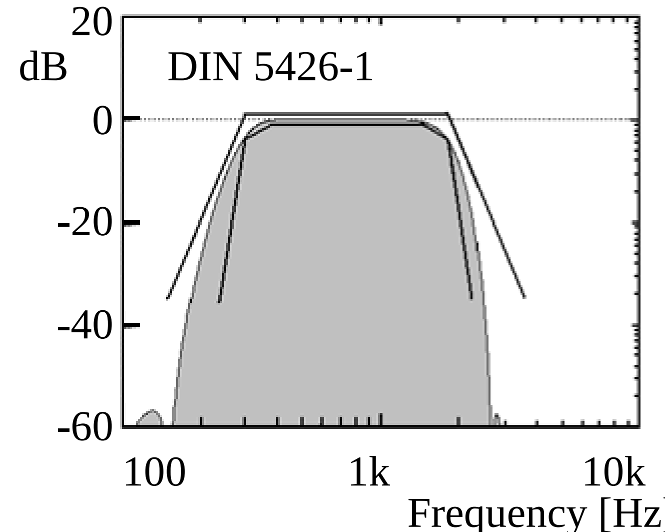
50 to 300 Hz

300 to 1800 Hz

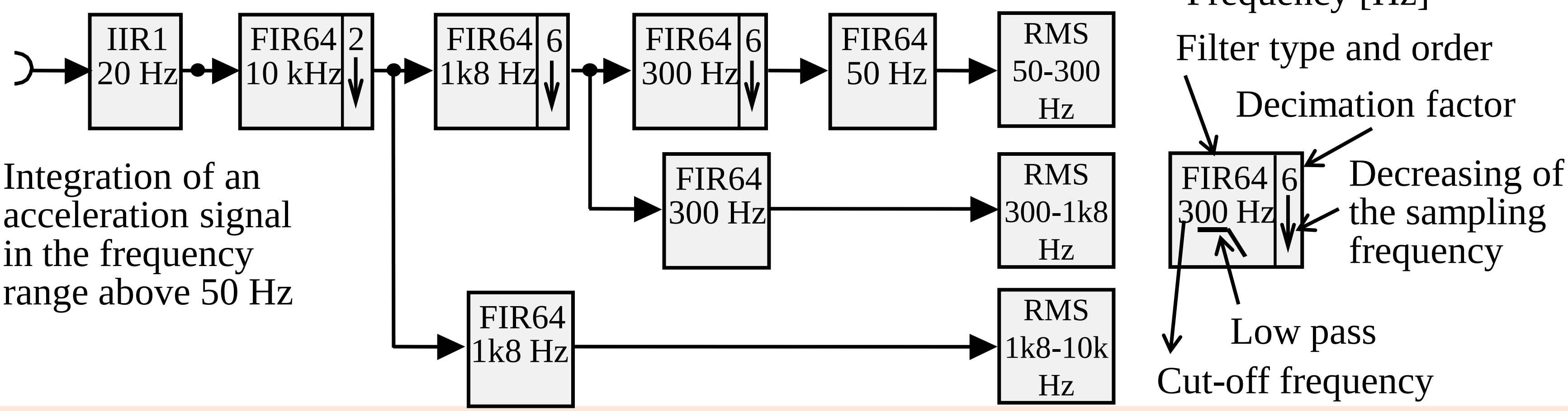
1k8 to 10k Hz

Design of three individual separate filters requires the FIR filter type of a large order, multirate filters seem to be an optimal solution

Frequency response of the passband filter



Input signal f_S = 44,1 kHz



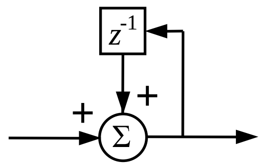
Integration of an acceleration signal in the frequency range above 50 Hz

Cascaded integrator-comb (CIC) filter

An integrator is simply a single-pole IIR filter with a unity feedback coefficient

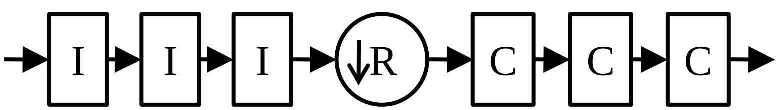
$$y_n = y_{n-1} + x_n$$

$$H_I(z) = \frac{1}{1 - z^{-1}}$$

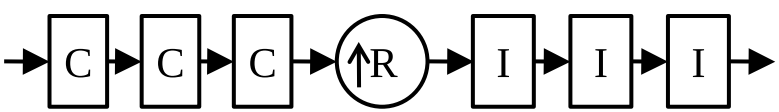


This integrator is known as an accumulator

Three stage (N = 3) decimating CIC filter



Three stage (N = 3) decimating CIC filter

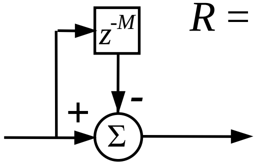


See [Donaldio]

A comb filter running at the high sampling rate, f_s, for a rate change of R is an odd symmetric FIR filter described by

$$y_n = x_n + x_{n-RM}$$

$$H_C(z) = 1 - z^{-RM}$$



where M is a design parameter and is called the differential delay (usually M = 1 or 2) and the positive integer R is designating a rate change.

Frequency characteristics

$$H(z) = H_I^N(z)_C^N H_C^N(z) = \frac{\left(1 - z^{-RM}\right)^N}{\left(1 - z^{-1}\right)^N} = \left(\sum_{k=0}^{RM-1} z^{-k}\right)^N$$

Gain of CIC decimators $G = (RM)^N$

Advantages:

- linear phase,
- utilize only delay and addition and subtraction

The CIC filter uses only fixed point math

Polyphase decomposition of sample sequences

Consider an arbitrary sequence of samples $x(n)$ with a Z-transform $X(z)$ given by

$$X(z)=\sum_{n=-\infty}^{+\infty} x(n) z^{-n}$$

The Z-transform of the polyphase decomposition of samples sequences is given by

$$X(z)=\sum_{k=0}^{M-1} z^{-k} X_k(z^M) \qquad X_k(z)=\sum_{n=-\infty}^{+\infty} x_k(n) z^{-n}=\sum_{n=-\infty}^{+\infty} x(Mn+k) z^{-n} \qquad 0\le k\le M-1$$

The subsequences $x_k(n)$ are called the polyphase components of the parent sequence $x(n)$

The functions $X_k(z)$, given by the Z-transforms of $x_k(n)$, are called the polyphase components of $X(z)$

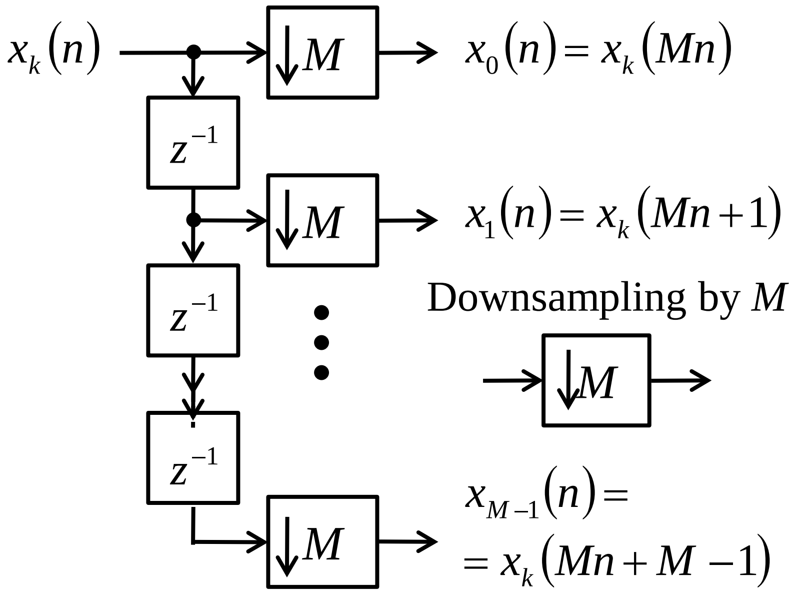
The relation between the subsequences $x_k(n)$ and the original sequence $x(n)$ is given by

$$x_k(n)=x_k(Mn+k), \quad 0\le k\le M-1$$

In matrix form we can write

$$X(z)=\begin{bmatrix}1 & z^{-1} & \dots & z^{-(M-1)}\end{bmatrix}\begin{bmatrix}X_0(z^M)\\ X_1(z^M)\\ \vdots \\ X_{M-1}(z^M)\end{bmatrix}$$

See [Mitra]



Polyphase decomposition of the transfer function

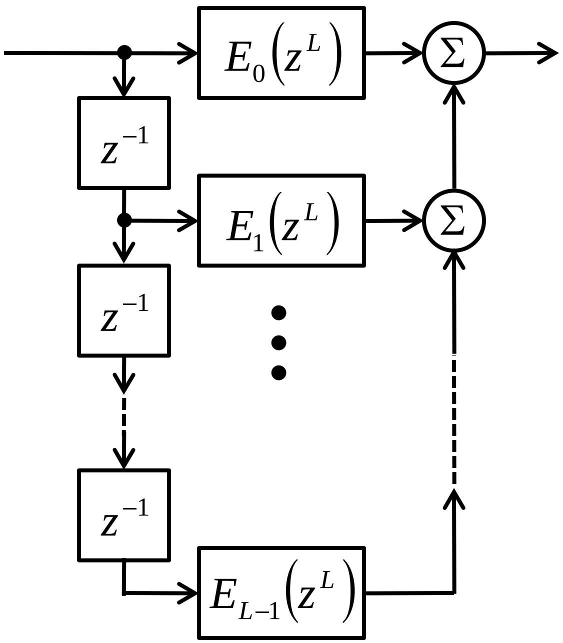
An L-branch polyphase decomposition of the transfer function of order N is of the form

$$H(z)=\sum_{m=0}^{L-1} z^{-m}E_m\left(z^L\right)$$

where

$$E_m(z)=\sum_{n=0}^{(N+1)/L} h(Ln+m)z^{-n}, \quad 0\le m\le L-1$$

with $h(n) = 0$ for $n > N$.



Linear phase FIR are characterized by a symmetric or antisymmetric impulse response.

$$h(n)=h(N-n) \qquad \text{or} \qquad h(n)=-h(N-n)$$

An example

$$H(z)=h(0)+h(1)z^{-1}+h(2)z^{-2}+h(2)z^{-3}+h(1)z^{-4}+h(0)z^{-5}$$

Polyphase decomposition

$$H(z)=E_0\left(z^3\right)+z^{-1}E_1\left(z^3\right)+z^{-2}E_2\left(z^3\right)$$

$$E_0(z)=h(0)+h(2)z^{-1}, \quad E_1(z)=h(1)+h(1)z^{-1}, \quad E_2(z)=h(2)+h(0)z^{-1}$$

See [Mitra]

Lth-band filters, half-band filter

Low pass filters with a transfer function that has certain zero-valued coefficients is called Nyquist or Lth-band filters. An Lth-band filter for L = 2 is called a half-band filter. The transfer function of a half-band filter is given by a formula

$$H(z) = \alpha + z^{-1}E_1\left(z^2\right)$$

The even samples of the impulse response are as follows

$$h(2n) = \begin{cases} \alpha, & n = 0 \\ 0, & \text{otherwise} \end{cases}$$

The sum of transfer functions

$$H(z) = \frac{1}{2} + z^{-1}E_1\left(z^2\right) \quad \text{and} \quad H(-z) = \frac{1}{2} - z^{-1}E_1\left((-z)^2\right)$$

gives

$$H(z) + H(-z) = 1$$

If $H(z)$ has real coefficients, then

$$H(-e^{j\omega T_S}) = H(e^{j(\pi+\omega T_S)}) = H^*(e^{j(\pi-\omega T_S)})$$

Hence

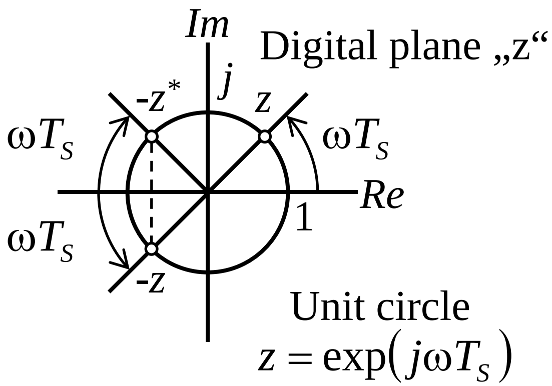
$$H(e^{j\omega T_S}) + H^*(e^{j(\pi-\omega T_S)}) = 1$$

The above equation implies that

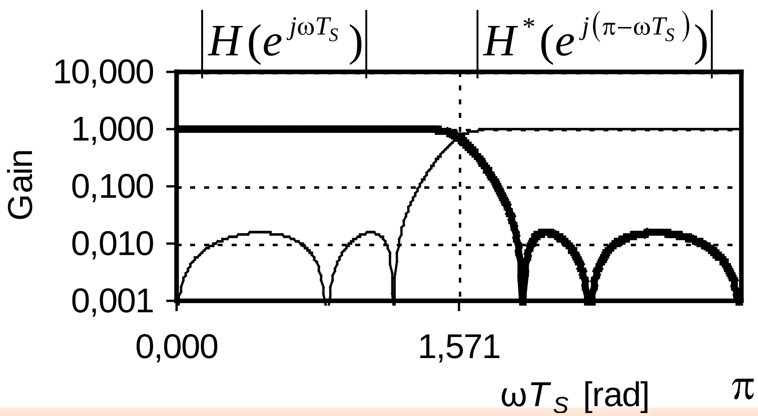
$$H(e^{j(\pi/2-\Omega)}) = H^*(e^{j(\pi/2+\Omega)})$$

$$\left|H(e^{j(\pi/2-\Omega)})\right| = \left|H(e^{j(\pi/2+\Omega)})\right|$$

See [Mitra]



Generally, if $H(z)$ has real coefficients, then $H(z) = H^*(z^*)$



Quadrature mixing in amplitude and phase demodulation

Let $x(t)$ be a signal in so-called envelope-and-phase form $x(t) = A(t)\sin(\omega_0 t + \Phi(t))$

The quadrature-carrier form of the signal is as follows …. $x(t) = I(t)\sin(\omega_0 t) + Q(t)\cos(\omega_0 t)$

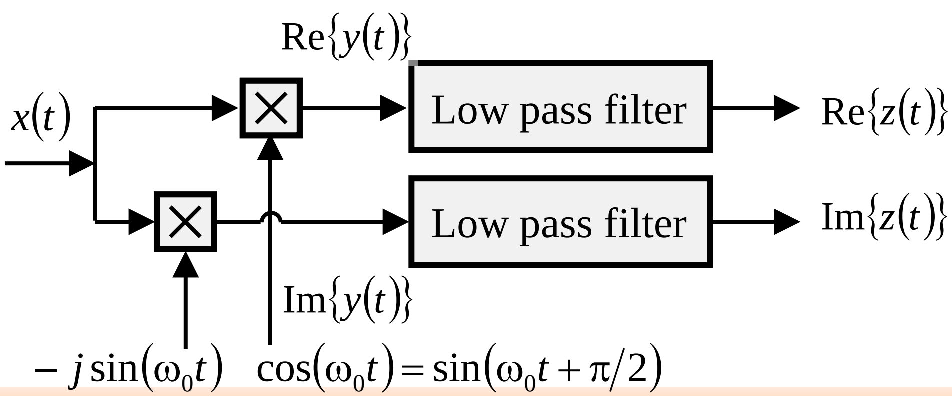
where $f_0 = \omega_0/2\pi$ is a carrier frequency and $I(t)$ and $Q(t)$ are modulation of a pure carrier wave $\sin(\omega_0 t)$ ………….. $I(t) = A(t)\sin(\Phi(t)) \quad Q(t) = A(t)\cos(\Phi(t))$

The component that is in phase with the original carrier $\sin(\omega_0 t)$ is referred to as the in-phase component while the out-of-phase component $\cos(\omega_0 t)$ is referred to as the quadrature component.

The quadrature mixing

The input signal $x(t)$ is transformed into a complex signal $y(t)$ and then filtered by a low pass filter

$$y(t) = A(t)\sin(\omega_0 t + \Phi(t))(\cos(\omega_0 t) - j\sin(\omega_0 t))$$



$$\mathrm{Re}\{y(t)\} = A(t)\sin(\omega_0 t + \Phi(t))\cos(\omega_0 t) =$$
$$= A(t)(\sin(\Phi(t)) + \sin(2\omega_0 t))/2$$
$$\mathrm{Im}\{y(t)\} = A(t)\sin(\omega_0 t + \Phi(t))\cos(\omega_0 t) =$$
$$= A(t)(\cos(\Phi(t)) - \cos(2\omega_0 t))/2$$

After low pass filtration

$$\mathrm{Re}\{z(t)\} = A(t)\sin(\Phi(t))/2$$
$$\mathrm{Im}\{z(t)\} = A(t)\cos(\Phi(t))/2$$

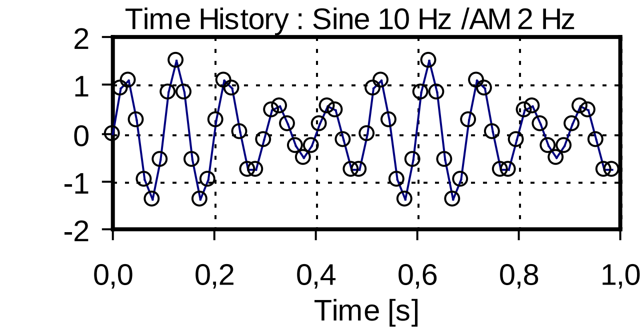
Envelope

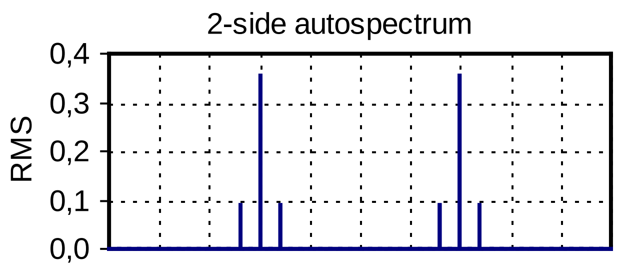
$$\mathrm{mag}(z(t)) = |z(t)| = A(t)/2 \Rightarrow A(t) = 2|z(t)|$$

Phase

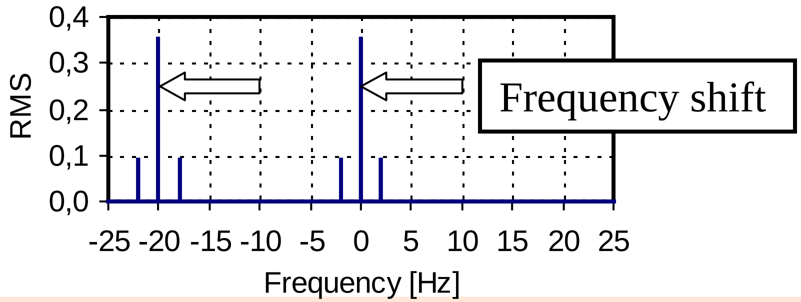
$$\mathrm{Re}\{z(t)\}/\mathrm{Im}\{z(t)\} = \tan(\Phi(t)) \Rightarrow$$
$$[\Phi(t)]_{\mathrm{WRAPPED}} = \arctan(\mathrm{Re}\{z(t)\}/\mathrm{Im}\{z(t)\})$$

Amplitude demodulation with the use of quadrature mixing





$$y(t) = x(t)\left(\cos(\omega_0 t) - j\sin(\omega_0 t)\right)$$



$$A(t) = 1 + 0.5\sin(\omega_M t)$$

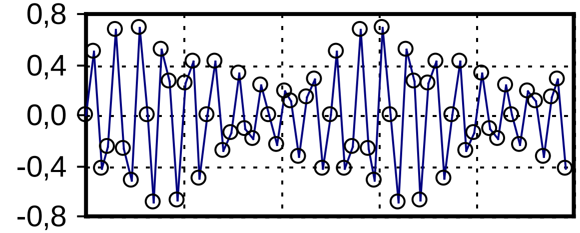
$$\Phi(t) = 0$$

Sampling frequency $f_S = 64$ Hz

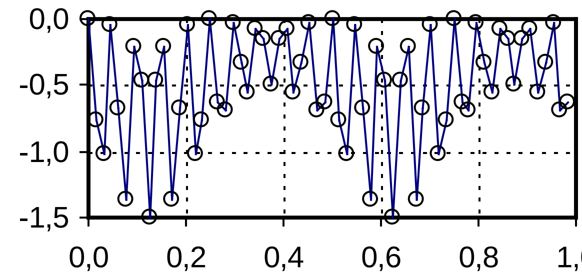
Carrying frequency $f_0 = 10$ Hz

Amplitude modulation index 0.5 frequency $f_M = 2$ Hz

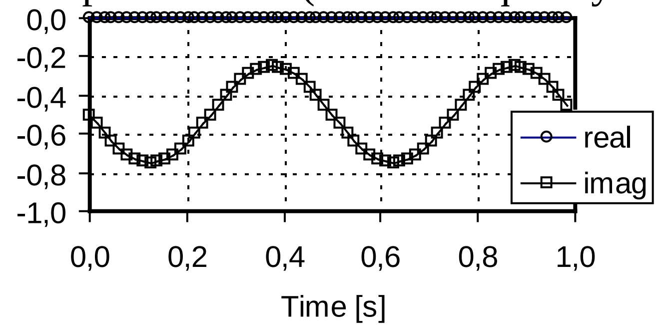
$$\mathrm{Re}\{y(t)\} = x(t)\cos(\omega_0 t)$$



$$\mathrm{Im}\{y(t)\} = -x(t)\sin(\omega_0 t)$$



Low pass filtered (cut-off frequency 5 Hz)



Goertzel algorithm

The algorithm solves a problem of identifying a frequency component in a signal (Dr. Gerald Goertzel, 1958) by focusing at specific, predetermined frequencies.

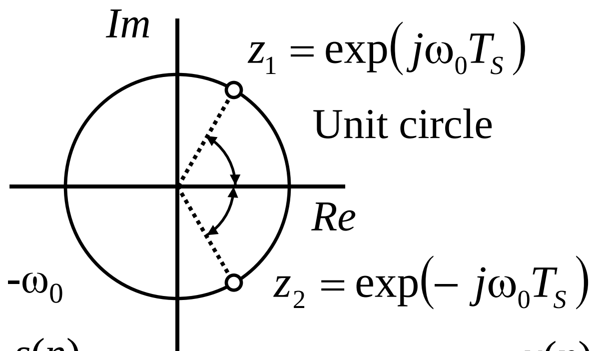
For a given sequence $x(n)$ the Goertzel algorithm computes a sequence $s(n)$, $n = \ldots,-2.-1,0,1,2,\ldots$

$$s(n) = x(n) + 2\cos(\omega_0 T_S)s(n-1) - s(n-2)$$

ω_0 – angular frequency,
T_S – sampling interval

The Z-transform of the previous difference equation results in

$$\frac{S(z)}{X(z)} = \frac{1}{1-2\cos(\omega_0 T_S)z^{-1}+z^{-2}} = \frac{1}{\left(1-e^{j\omega_0 T_S}z^{-1}\right)\left(1-e^{-j\omega_0 T_S}z^{-1}\right)}$$



The poles of the transfer function lies on the unit circle. The frequency response tends to infinity for $\omega = +\omega_0$ and $\omega = -\omega_0$

Let the sequence $s(n)$ be filtered by a FIR filter with the zero at $\omega = -\omega_0$

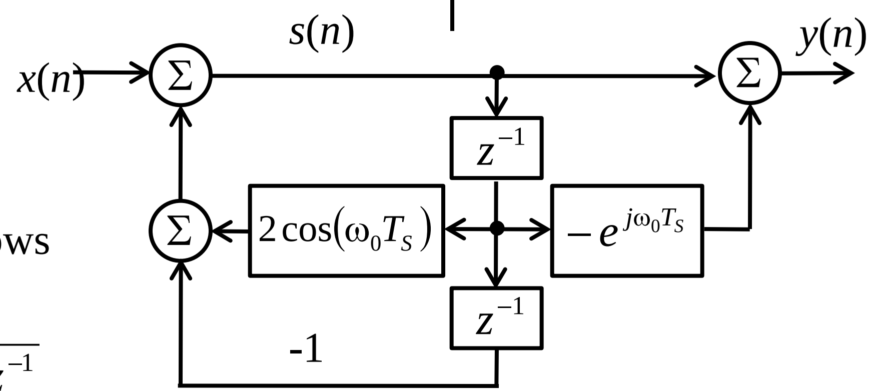
$$\frac{Y(z)}{S(z)} = 1-\exp(-j\omega_0 T_S)z^{-1}$$

The corresponding difference equation

$$y(n) = s(n) - \exp(-j\omega_0 T_S)s(n-1)$$

The transfer function relating $Y(z)$ to $X(z)$ is as follows

$$\frac{Y(z)}{X(z)} = \frac{1-e^{-j\omega_0 T_S}z^{-1}}{\left(1-e^{j\omega_0 T_S}z^{-1}\right)\left(1-e^{-j\omega_0 T_S}z^{-1}\right)} = \frac{1}{1-e^{j\omega_0 T_S}z^{-1}}$$



Goertzel algorithm – cont'd

The time-domain equivalent of the previous transfer function

$$y(n)=x(n)+e^{j\omega_0 T_S}y(n-1)=\sum_{k=-\infty}^{n}x(k)e^{j\omega_0 T_S(n-k)}=e^{j\omega T_S n}\sum_{k=-\infty}^{n}x(k)e^{-j\omega_0 T_S k}$$

Assuming $x(k)$ for all $k<0$, we obtain

$$y(n)=e^{j\omega_0 T_S n}\sum_{k=0}^{n}x(k)e^{-j\omega_0 T_S k}=e^{j\omega_0 T_S n}X(\omega_0) \qquad \left(X(\omega_0)=y(n)e^{-j\omega_0 T_S n}\right)$$

Except of the scale factor $\exp(+j\omega T_S n)$ the sample $y(n)$ depends on DFT of the $(n+1)$ samples of $x(n)$.

Evaluation of $y(n)$ requires only the last two samples of $s(n)$ (the output of the FIR filter), which can be used to compute the DFT of $x(n)$ corresponding to $X(\omega_0)$

$$X(\omega_0)=\left(s(n-1)-e^{-j\omega_0 T_S}s(n-2)\right)e^{j\omega_0 T_S n}=e^{j\omega_0 T_S n}s(n-1)-e^{j\omega_0 T_S(n-1)}s(n-2)$$

where

$$s(n)=x(n)+2\cos(\omega_0 T_S)s(n-1)-s(n-2)$$

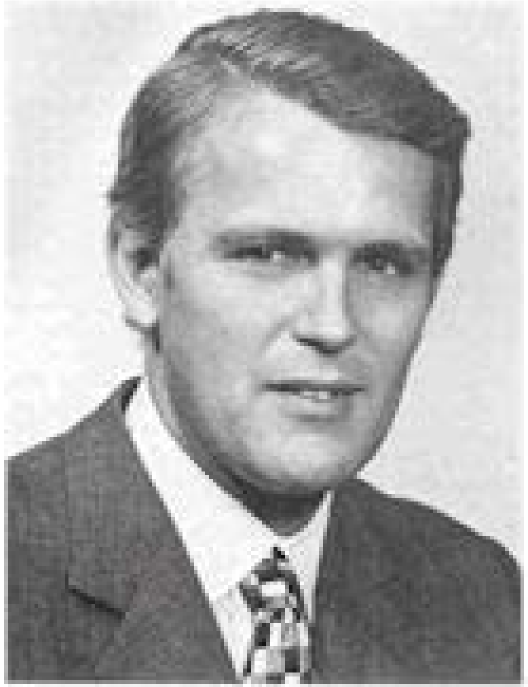
The algorithm starts assuming $s(-1)=s(-2)=0$.

The power at the frequency ω_0 can be computed using the formula

$$X(\omega_0)X^*(\omega_0)=(s(n-2))^2+(s(n-1))^2-2\cos(\omega_0 T_S)s(n-2)s(n-1)$$

See http://en.wikipedia.org/wiki/Goertzel_algorithm

Rudolf Emil Kalman, born on May 19, 1930, in Budapest, Hungary, is a Hungarian-American electrical engineer, mathematical system theorist, and college professor, who was educated in the United States, and has done most of his work there.



KALMAN FILTER

An example of a recursive filter

Assume that we have a system whose one-dimensional state x we can measure at successive steps: $x(1)$, $x(2)$, …, $x(k)$. The problem is to compute the average $\mu(k)$ of the time series given k samples. The solution is

$$\mu(k) = \frac{1}{k}\sum_{1}^{k} x(i)$$

Adding a new measurement $x(k+1)$ the new average value is obtained

$$\mu(k+1) = \frac{1}{k+1}\sum_{1}^{k+1} x(i) = \frac{k}{k+1}\left(\frac{1}{k}\sum_{1}^{k} x(i) + \frac{1}{k}x(k+1)\right)$$

and so, $\mu(k+1)$ can be written

$$\mu(k+1) = \frac{k}{k+1}\mu(k) + \frac{1}{k+1}x(k+1) = \mu(k) + K\left(x(k+1) - \mu(k)\right)$$

where $K = 1/(k+1)$ is a gain factor. The new average $\mu(k+1)$ is a weighted average of the old estimate $\mu(k)$ and the new value of $x(k+1)$. If k is approaching to infinity, the gain factor tends to zero.

We can also recalculate recursively the variance of the time series. Given k samples, the variance is computed by

$$\sigma^2(k) = \frac{1}{k}\sum_{1}^{k}\left(x(i) - \mu(k)\right)^2$$

If a new sample $x(k+1)$ is measured, the new variance adopts to the value

$$\sigma^2(k+1) = \frac{1}{k+1}\sum_{1}^{k+1}\left(x(i) - \mu(k+1)\right)^2 = \frac{1}{k+1}\sum_{1}^{k+1}\left(x(i) - \mu(k) - K\left(x(k+1) - \mu(k)\right)\right)^2 = \ldots =$$

$$= (1-K)\left(\sigma^2(k) + K\left(x(k+1) - \mu(k)\right)^2\right)$$

The gain factor, independend on k, controls the value of the variance and enables to follow a slow variation of the average μ in time.

Conditional expected value or mean value

Let a set $\mathbf{X}(n)$ of measurements be given by a sequence of samples $x(k)$

$$\mathbf{X}(n) = \{x(i), 1 \le i \le n\}$$

Conditional probability density function of a discrete random variable $\xi(k)$ conditioned on the set of measurements $\mathbf{X}(n)$ (in fact a random vector $\mathbf{\Xi}(n) = \{\xi(i) , 1 <= i <= n\}$) is given by

$$p(x(n+1)|\mathbf{X}(n)) = \frac{P(\xi(n+1) = x(n+1) \cap \xi(1) = x(1) \cap \ldots \cap \xi(n) = x(n))}{P(\xi(1) = x(1) \cap \ldots \cap \xi(n) = x(n))} =$$

$$= \frac{P(\xi(k) = x(k) \cap \mathbf{\Xi}(n) = \mathbf{X}(n))}{P(\mathbf{\Xi}(n) = \mathbf{X}(n))}$$

where the probability of the occurrence of $\mathbf{X}(n)$ is a positive value, i.e. $P(\mathbf{X}(n)) > 0$. Similarly for continuous random variable can be defined $p(x(n+1)| \mathbf{X}(n))$. Instead of $\xi = x$ the random variable ξ belongs to the interval $\mathbf{I}(x) = \{x < \xi <= x + \Delta x\}$

$$p(x(n+1)|\mathbf{X}(n))\Delta x = \frac{P(\xi(n+1) \in \mathbf{I}(x(n+1)) \cap \xi(1) \in \mathbf{I}(x(1)) \cap \ldots \cap \xi(n) \in \mathbf{I}(x(n)))}{P(\xi(1) \in \mathbf{I}(x(1)) \cap \ldots \cap \xi(n) \in \mathbf{I}(x(n)))}$$

The conditional expected value, or mean value, of a continuous random variable $\xi(n+1)$ conditioned on the set of measurements $\mathbf{X}(n)$ is defined as

$$\mathrm{E}\{x|\mathbf{X}(n)\} = \int_{-\infty}^{+\infty} x\, p(x|\mathbf{X}(n))\,\mathrm{d}\,x$$

Minimum Mean-Square Error

The mean value of the squared difference between the random variable x and the estimate of x (mean-square error - MSE) is given by

$$E\left\{(x-\hat{x})^2\right\}=E\left\{\left((x-E\{x\})+(E\{x\}-\hat{x})\right)^2\right\}=E\left\{(x-E\{x\})^2-2(x-E\{x\})(E\{x\}-\hat{x})+(E\{x\}-\hat{x})^2\right\}=$$
$$=E\left\{(x-E\{x\})^2\right\}-2(E\{x\}-\hat{x})E\left\{(x-E\{x\})\right\}+E\left\{(E\{x\}-\hat{x})^2\right\}=\sigma_x^2-0+(E\{x\}-\hat{x})^2$$

According to the fundamental theorem of the estimation theory the minimum value of MSE is reached if the estimate of x is equal to the mean value of x

$$E\{x\}-\hat{x}=0 \quad \Rightarrow \quad \hat{x}^{MMSE}=E\{x\} \quad \text{and} \quad \sigma_x^2=E\left\{(x-\hat{x})^2\right\}$$

The variance of MMSE is the same as the variance of the random variable itself. Both the definitions, MMSE and the variance of MSE, may be written as

$$\hat{x}^{MMSE}=E\{x\}=\int_{-\infty}^{+\infty} x\, p(x)\,\mathrm{d}\,x \qquad P^{MMSE}=\sigma_x^2=\mathrm{E}\left\{(x-\hat{x})^2\right\}=\int_{-\infty}^{+\infty}(x-\hat{x})^2\, p(x)\,\mathrm{d}\,x$$

The estimate of x conditioned on $X(n)$ and the minimum value of the mean square error

$$\hat{x}^{MMSE}=E\{x\,|\,\mathbf{X}\}=\int_{-\infty}^{+\infty} x\, p(x\,|\,\mathbf{X})dx \qquad P^{MMSE}=\sigma_x^2=\mathrm{E}\left\{(x-\hat{x})^2\,|\,\mathbf{X}(n)\right\}=\int_{-\infty}^{+\infty}(x-\hat{x})^2\, p(x\,|\,\mathbf{X}(n))\,\mathrm{d}\,x$$

The minimum of the MSE variance can be found by taking the derivative of the function with respect to x and setting that value to 0.

Kalman filter - Process and measurement models

The Kalman filter addresses the general problem of estimating the $n \times 1$ state vector $\mathbf{x}$ of a discrete time process that is governed by difference process equation

$$\mathbf{x}(k) = \mathbf{A}(k)\mathbf{x}(k-1) + \mathbf{B}(k)\mathbf{u}(k) + \mathbf{v}_1(k)$$

with the measurement (observation) vector $\mathbf{y}$ that is defined by a measurement equation, describing the observation as

$$\mathbf{y}(k) = \mathbf{H}(k)\mathbf{x}(k) + \mathbf{v}_2(k)$$

where random variables $\mathbf{v}_1$ a $\mathbf{v}_2$ represent the process and measurement noise (respectively). It is assumed that the random variables are independent of each other and with normal probability distribution

$$p(\mathbf{v}_1) \sim N(\mathbf{0}, \mathbf{Q}(k)) \qquad p(\mathbf{v}_2) \sim N(\mathbf{0}, \mathbf{R}(k))$$

The correlation matrices are defined as

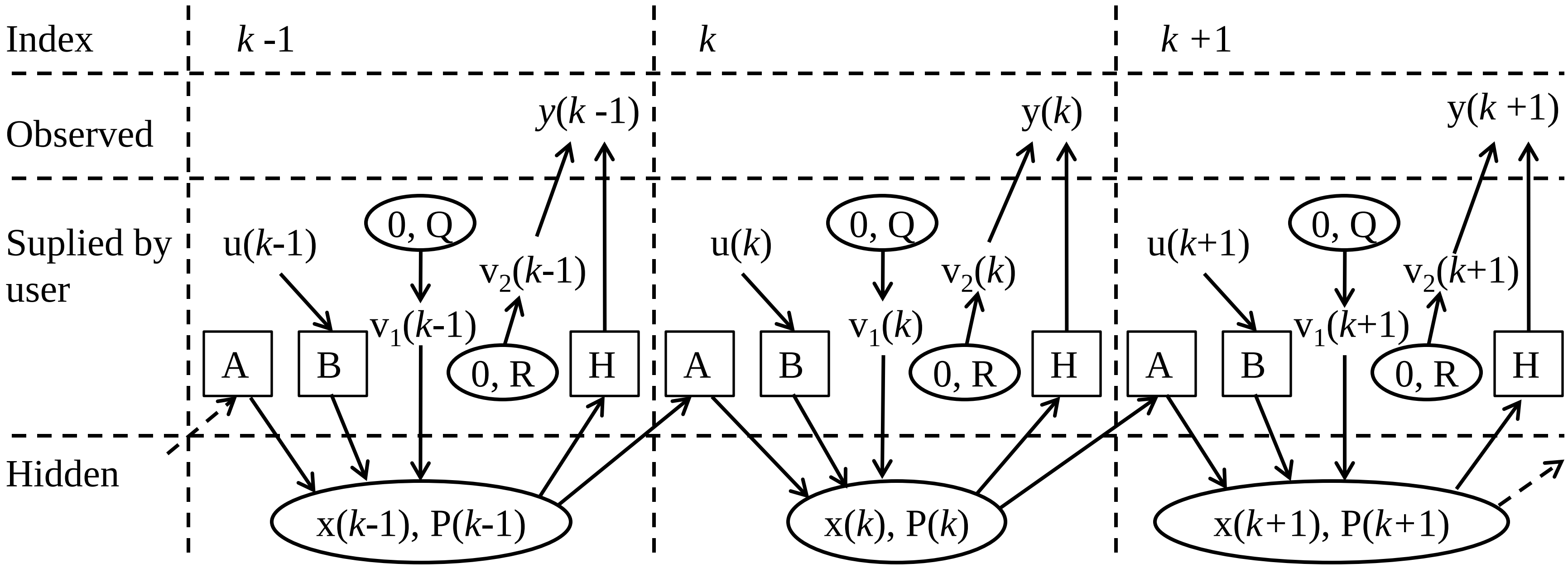
$$E\{\mathbf{v}_1(n)\mathbf{v}_1^T(k)\} = \begin{cases} \mathbf{Q}(n), & n = k \\ \mathbf{0}, & n \neq k \end{cases} \qquad E\{\mathbf{v}_2(n)\mathbf{v}_2^T(k)\} = \begin{cases} \mathbf{R}(n), & n = k \\ \mathbf{0}, & n \neq k \end{cases}$$

$$E\{\mathbf{v}_1(n)\mathbf{v}_2^T(k)\} = \mathbf{0} \quad \text{for all } n \text{ and } k$$

The $n \times n$ matrix $\mathbf{A}$ is the state transition model which is applied to the previous state $\mathbf{x}(k-1)$, the matrix $\mathbf{B}$ is the control-input model which is applied to the control vector $\mathbf{u}(k)$ and the $m \times n$ matrix $\mathbf{H}$ is the observation model which maps the true state space into the observed space.

Kalman filter - Evolution of states in time

The evolution of states in time is shown in the following diagram



The various matrices are constant with time, and thus the designation of time steps is dropped

The Kalman filter may be considered as a recursive estimator. This means that only the estimated state from the previous time step and the current measurement are needed to compute the estimate for the current state.

Kalman filter - Basic notation

Let a special notation be introduced. The term $\hat{\mathbf{x}}(n|m)$ represents the estimate of $\mathbf{x}$ at time n given observations up to, and including at time m.

Let a set $\mathbf{Y}(k)$ of measurements (observations) be given by a sequence of samples $\mathbf{y}(k)$

$$\mathbf{Y}(k) = \{y(i), 1 \le i \le k\}$$

An *a priori* state estimate $\hat{\mathbf{x}}(k|k-1)$ at step k is given by knowledge of the process prior to step k and an *a posteriory* state estimate $\hat{\mathbf{x}}(k|k)$ at step k is given by measurement $\mathbf{y}(k)$ at step k

For this set $\mathbf{Y}(n)$ of measurements it is possible to define

$$\hat{\mathbf{x}}(k|k-1) = E\{\mathbf{x}(k) \mid \mathbf{Y}(k-1)\}$$ The *a priori* state estimate

$$\hat{\mathbf{x}}(k|k) = E\{\mathbf{x}(k) \mid \mathbf{Y}(k)\}$$ The *a posteriori* state estimate

The *a priori* estimate error covariance is given by

$$\mathbf{P}(k|k-1) = \operatorname{cov}(\mathbf{x}(k) - \hat{\mathbf{x}}(k|k-1)) = E\left\{(\mathbf{x}(k) - \hat{\mathbf{x}}(k|k-1))(\mathbf{x}(k) - \hat{\mathbf{x}}(k|k-1))^T\right\}$$

We can then define the *a posteriori* error covariance matrix (a measure of the estimated accuracy of the state estimate) as

$$\mathbf{P}(k|k) = \operatorname{cov}(\mathbf{x}(k) - \hat{\mathbf{x}}(k|k)) = E\left\{(\mathbf{x}(k) - \hat{\mathbf{x}}(k|k))(\mathbf{x}(k) - \hat{\mathbf{x}}(k|k))^T\right\}$$

The covariance matrices are symmetric $\mathbf{P}^T(k \mid k) = \mathbf{P}(k \mid k)$, $\mathbf{P}^T(k \mid k-1) = \mathbf{P}(k \mid k-1)$.

Time and measurement update

The process model updates the estimate of the next value of x

$$P(\mathbf{x}(k)\,|\,\mathbf{y}(1),\ldots,\mathbf{y}(k)) \rightarrow P(\mathbf{x}(k+1)\,|\,\mathbf{y}(1),\ldots,\mathbf{y}(k))$$

As it was stated before the estimate for the minimum value of MSE is as follows

$$\hat{\mathbf{x}}(k+1|k) = E\{\mathbf{A}(k)\mathbf{x}(k) + \mathbf{B}(k)\mathbf{u}(k) + \mathbf{v}_1(k)\} = \mathbf{A}(k)E\{\mathbf{x}(k)\} + \mathbf{B}(k)\mathbf{u}(k) + 0 =$$

$$= \mathbf{A}(k)\hat{\mathbf{x}}(k|k) + \mathbf{B}(k)\mathbf{u}(k)$$

Note, that the estimation of the state vector $\mathbf{x}$ at step $k+1$ does not reflect the new observation at step $k+1$.

The new measurement (observation) updates the probability function parameters

$$P(\mathbf{x}(k)\,|\,\mathbf{y}(1),\ldots,\mathbf{y}(k-1)) \rightarrow P(\mathbf{x}(k)\,|\,\mathbf{y}(1),\ldots,\mathbf{y}(k))$$

The *a priori* estimate error covariance (taken before the new observation) is updated

$$\mathbf{P}(k|k-1) = \operatorname{cov}(\mathbf{x}(k) - \hat{\mathbf{x}}(k|k-1)) =$$

$$= \operatorname{cov}(\mathbf{A}(k-1)\mathbf{x}(k-1) + \mathbf{B}(k-1)\mathbf{u}(k-1) + \mathbf{v}_1(k-1) - \hat{\mathbf{x}}(k|k-1)) =$$

$$= \mathbf{A}(k-1)\operatorname{cov}(\mathbf{x}(k-1))\mathbf{A}^T(k-1) + \operatorname{cov}(\mathbf{v}_1(k-1)) =$$

$$= \mathbf{A}(k-1)\mathbf{P}(k-1|k-1)\mathbf{A}^T(k-1) + \mathbf{Q}(k-1)$$

After substituting $k+1 \rightarrow k$ it is obtained

$$\mathbf{P}(k+1|k) = \mathbf{A}(k)\mathbf{P}(k|k)\mathbf{A}^T(k) + \mathbf{Q}(k)$$

Innovation of measurement residual

The difference between an actual measurement $\mathbf{y}(k)$ and a measurement prediction $\hat{\mathbf{y}}(k|k-1)$ at step k is given by knowledge of the process prior to step k is called as an innovation of measurement residual

$$\alpha(k) = \mathbf{y}(k) - \hat{\mathbf{y}}(k|k-1)$$

where the estimate of the measurement prediction is as follows

$$\hat{\mathbf{y}}(k|k-1) = E\{\mathbf{y}(k)|\mathbf{Y}(k-1)\} = E\{\mathbf{H}(k)\mathbf{x}(k) + \mathbf{v}_2(k)|\mathbf{Y}(k-1)\} = \mathbf{H}(k)\hat{\mathbf{x}}(k|k-1)$$

The innovation $\alpha(k)$ has several important properties

1) The innovation $\alpha(k)$, associated with the observed random variable $y(k)$ is orthogonal to the past observations $y(1)$, $y(2)$, $y(3)$, …, $y(k-1)$, as shown

$$E\{\alpha(k)y(i)\} = 0, \quad 1 \le i \le k-1$$

2) The innovation $\alpha(1)$, $\alpha(1)$, $\alpha(2)$, …, $\alpha(k)$ are orthogonal to each other, as shown

$$E\{\alpha(k)\alpha(i)\} = 0, \quad 1 \le i \le k-1$$ The innovation process is white.

3) There is one-to-one correspondence between the observed data [$y(1)$, $y(2)$, $y(3)$, …, $y(k)$] and the innovations [$\alpha(1)$, $\alpha(2)$, $\alpha(3)$, …, $\alpha(k)$], in that the one sequence may be obtained from the other by means the causal and causally invertible filter without any loss of information.

Using the Gram/Schmidt orthogonalization procedure it is possible to prove this property. The procedure assumes that the observations [$y(1)$, $y(2)$, $y(3)$, …, $y(k)$] are linearly independent in an algebraic sense.

Innovation of measurement residual – cont'd 1

We first put

$$\alpha(1) = y(1)$$

Next we put

$$\alpha(2) = y(2) + a_{1,1} y(1)$$

The coefficient $a_{1,1}$ is chosen such that the innovation α(1) and α(1) are orthogonal

$$E\{\alpha(2)\alpha(1)\} = \{(y(2) + a_{1,1} y(1))y(1)\} = 0 \quad \Rightarrow \quad a_{1,1} = -E\{y(2)y(1)\}/E\{y(1)y(1)\}$$

Next we put

$$\alpha(3) = y(3) + a_{2,1} y(2) + a_{2,2} y(1)$$

…………..

In general, we may transform of the observed data [$y(1)$, $y(2)$, $y(3)$, … , $y(k)$] and the innovations [α(1) , α(2) , α(3) , … , α(k)] by writing

$$\begin{bmatrix} \alpha(1) \\ \alpha(2) \\ \\ \alpha(k) \end{bmatrix} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ a_{1,1} & 1 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ a_{k-1,k-1} & a_{k-1,k-2} & \dots & 1 \end{bmatrix} \begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(k) \end{bmatrix}$$

The matrix is nonsingular since its determinant is equal to one. The transformation is therefore reversible.

Innovation of measurement residual – cont’d 2

Now we turn attention to the *a posteriori* state estimate $\hat{\mathbf{x}}(k|k)$. If $\mathbf{x}(k)$ is a vector n x 1 then the diagonal elements of the covariance matrix $\mathbf{P}(k|k)$ are variances of the difference between the true value of the components of the state vector $\mathbf{x}$ and its *a posteriori* estimate given the observed data $[y(1), y(2), y(3), \dots, y(k)]$.

$$\mathbf{P}(k|k) = \begin{bmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_n^2 \end{bmatrix} \quad \Rightarrow \quad \mathrm{tr}(\mathbf{P}(k|k)) = \sigma_1^2 + \dots + \sigma_n^2$$

It is obvious that the smaller the trace of the matrix $\mathbf{P}(k|k)$, the more accurate (less variance) estimate of the *a posteriori* state estimate $\hat{\mathbf{x}}(k|k)$ given the observed data $[y(1), y(2), y(3), \dots, y(k)]$.

Let the *a posteriori* state estimate to be the minimum mean-square estimation of $\mathbf{x}(k)$ given the observed data $[\alpha(1), \alpha(2), \alpha(3), \dots, \alpha(k)]$ as well due to the correspondence between the observed data $[y(1), y(2), y(3), \dots, y(k)]$ and the innovations $[\alpha(1), \alpha(2), \alpha(3), \dots, \alpha(k)]$.

The minimum mean-square estimation of $\mathbf{x}(k)$ given the observed data may be defined as a linear combination of the innovations $[\alpha(1), \alpha(2), \alpha(3), \dots, \alpha(k)]$

$$\hat{\mathbf{x}}(k|k) = \sum_{1}^{k} b(i)\alpha(i)$$

As the innovation are orthogonal to each other the coefficient $b(i)$ may be determined by

$$b(i) = \frac{E\{x(k)\alpha(i)\}}{E\{\alpha(i)\alpha(i)\}}, \quad 1 \le i \le k$$

Innovation of measurement residual – cont'd 3

The recursive form for the linear combination of innovations

$$\hat{\mathbf{x}}(k|k) = \sum_{1}^{k-1} b(i)\alpha(i) + b(k)\alpha(k)$$
$$= \hat{\mathbf{x}}(k-1|k-1) + b(k)\alpha(k)$$

where

$$b(k) = \frac{E\{x(k)\alpha(k)\}}{E\{\alpha(k)\alpha(k)\}}$$

This is a reason that the *a posteriori* state estimate $\hat{\mathbf{x}}(k|k)$ can be computed as a linear combination of an *a priori* estimate $\hat{\mathbf{x}}(k|k-1)$ and the innovation of measurement residual as shown below

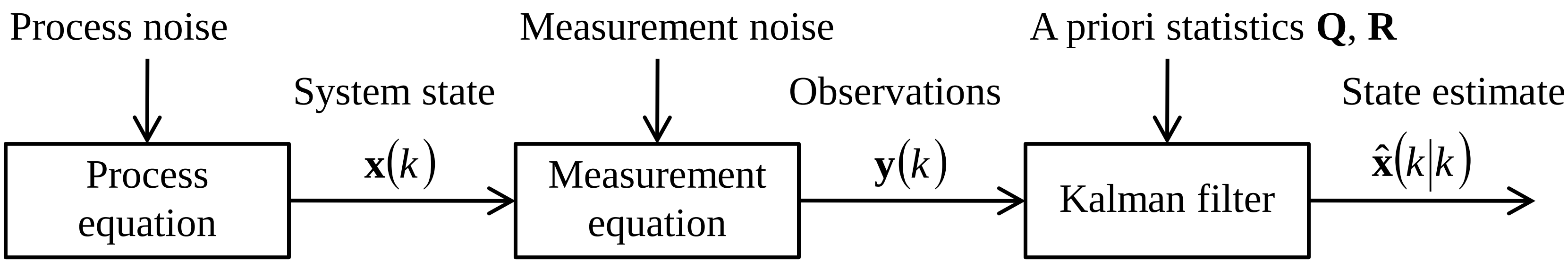
$$\hat{\mathbf{x}}(k|k) = \hat{\mathbf{x}}(k|k-1) + \mathbf{K}(k)\hat{\boldsymbol{\alpha}}(k) =$$
$$= \hat{\mathbf{x}}(k|k-1) + \mathbf{K}(k)\left(\mathbf{y}(k) - \mathbf{H}(k)\hat{\mathbf{x}}(k|k-1)\right)$$

where $\mathbf{K}(k)$ is a Kalman gain replacing the coefficient $b(k)$.

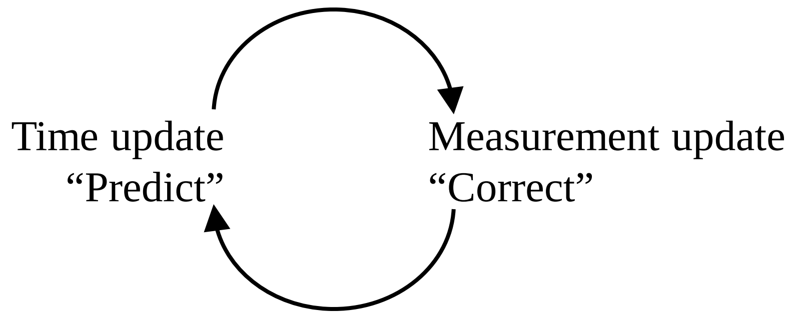
The Kalman gain is a tuning button which adjust the process of filtration to trace variation of the system state to minimize the difference between the true state and its estimation.

The optimal Kalman gain is given by solving of the matrix equation resulting from the zero valu of the first partial derivative with respect to the Kalman gain $\frac{\partial\, \mathrm{tr}\left(\mathbf{P}(k|k)\right)}{\partial \mathbf{K}(k)} = 0$

Estimator model



The time update equations can also be considered as *predictor equations*, while the measurement update equations can be considered as *corrector equations*. The final estimation algorithm resembles that of a *predictor-corrector algorithm for solving* numerical problems as shown below



Kalman filter – cont'd 1

The Kalman filter has two distinct phases: Predict and Update. The predict phase uses the state estimate from the previous timestep to produce an estimate of the state at the current timestep. This predicted state estimate is also known as the a priori state estimate because, although it is an estimate of the state at the current timestep, it does not include observation information from the current timestep. In the update phase, the current a priori prediction is combined with current observation information to refine the state estimate. This improved estimate is termed the a posteriori state estimate.

Predict phase

$$\hat{\mathbf{x}}(k|k-1) = \mathbf{A}(k)\hat{\mathbf{x}}(k-1|k-1) + \mathbf{B}(k)\mathbf{u}(k)$$ Predicted (a priori) state

$$\mathbf{P}(k|k-1) = \mathbf{A}(k)\mathbf{P}(k-1|k-1)\mathbf{A}^T(k) + \mathbf{Q}(k)$$ Predicted (a priori) estimate covariance

Update phase

$$\boldsymbol{\alpha}(k) = \mathbf{y}(k) - \mathbf{H}(k)\hat{\mathbf{x}}(k|k-1)$$ Innovation of measurement residual

$$\mathbf{S}(k) = \mathbf{H}(k)\mathbf{P}(k|k-1)\mathbf{H}^T(k) + \mathbf{R}(k)$$ Innovation (or residual) covariance $\mathbf{S}(k) = \mathrm{cov}(\boldsymbol{\alpha}(k))$

$$\mathbf{K}(k) = \mathbf{P}(k|k-1)\mathbf{H}^T(k)\mathbf{S}^{-1}(k)$$ Optimal Kalman gain

$$\hat{\mathbf{x}}(k|k) = \hat{\mathbf{x}}(k|k-1) + \mathbf{K}(k)\boldsymbol{\alpha}(k)$$ Updated (a posteriori) state estimate

$$\mathbf{P}(k|k) = (\mathbf{E} - \mathbf{K}(k)\mathbf{H}(k))\mathbf{P}(k|k-1)$$ Updated (a posteriori) estimate covariance (**E** is an identity or unit matrix)

http://en.wikipedia.org/wiki/Kalman_filter

Derivations – a posteriori estimate covariance matrix

If the model is accurate, and the values for $\hat{\mathbf{x}}(0|0)$ and $\mathbf{P}(0|0)$ accurately reflect the distribution of the initial state values, then the following invariants are preserved: (all estimates have mean error zero)

$$E\{\mathbf{x}(k)-\hat{\mathbf{x}}(k|k)\}=E\{\mathbf{x}(k)-\hat{\mathbf{x}}(k|k-1)\}=0$$

$$E\{\tilde{\mathbf{z}}(k)\}=0$$

where $E\{\ldots\}$ is the expected (mean) value.

Deriving the a posteriori estimate covariance matrix

$$\mathbf{P}(k|k)=\operatorname{cov}(\mathbf{x}(k)-\hat{\mathbf{x}}(k|k))$$

$$\mathbf{P}(k|k)=\operatorname{cov}(\mathbf{x}(k)-(\hat{\mathbf{x}}(k|k-1)-\mathbf{K}(k)\boldsymbol{\alpha}(k)))$$

$$\mathbf{P}(k|k)=\operatorname{cov}(\mathbf{x}(k)-(\hat{\mathbf{x}}(k|k-1)-\mathbf{K}(k)(\mathbf{y}(k)-\mathbf{H}(k)\hat{\mathbf{x}}(k|k-1))))$$

$$\mathbf{P}(k|k)=\operatorname{cov}(\mathbf{x}(k)-(\hat{\mathbf{x}}(k|k-1)-\mathbf{K}(k)(\mathbf{H}(k)\mathbf{x}(k)+\mathbf{v}_2(k)-\mathbf{H}(k)\hat{\mathbf{x}}(k|k-1))))$$

$$\mathbf{P}(k|k)=\operatorname{cov}((\mathbf{E}-\mathbf{K}(k)\mathbf{H}(k))(\mathbf{x}(k)-\hat{\mathbf{x}}(k|k-1))-\mathbf{K}(k)\mathbf{v}_2(k))$$

$$\mathbf{P}(k|k)=\operatorname{cov}((\mathbf{E}-\mathbf{K}(k)\mathbf{H}(k))(\mathbf{x}(k)-\hat{\mathbf{x}}(k|k-1)))+\operatorname{cov}(\mathbf{K}(k)\mathbf{v}_2(k))$$

$$\mathbf{P}(k|k)=(\mathbf{E}-\mathbf{K}(k)\mathbf{H}(k))\operatorname{cov}((\mathbf{x}(k)-\hat{\mathbf{x}}(k|k-1)))(\mathbf{E}-\mathbf{K}(k)\mathbf{H}(k))^T+\mathbf{K}(k)\operatorname{cov}(\mathbf{v}_2(k))\mathbf{K}^T(k)$$

$$\mathbf{P}(k|k)=(\mathbf{E}-\mathbf{K}(k)\mathbf{H}(k))\mathbf{P}(k|k-1)(\mathbf{E}-\mathbf{K}(k)\mathbf{H}(k))^T+\mathbf{K}(k)\mathbf{R}(k)\mathbf{K}^T(k)$$

http://en.wikipedia.org/wiki/Kalman_filter

Derivations – the Kalman gain

The Kalman filter is a minimum mean-square error estimator of the error in the a posteriori state estimation, that an objective function is as follows

$$E\left\{\left|\mathbf{x}(k)-\hat{\mathbf{x}}(k|k)\right|^2\right\}\to \min$$

This is equivalent to minimizing the trace of the a posteriori estimate covariance matrix $\mathbf{P}(k \mid k)$. By expanding out the terms in the equation above and collecting, we obtain

$$\begin{aligned}\mathbf{P}(k|k) &= \mathbf{P}(k|k-1)-\mathbf{K}(k)\mathbf{H}(k)\mathbf{P}(k|k-1)-\mathbf{P}(k|k-1)\mathbf{H}^T(k)\mathbf{K}^T(k)+\\ &\quad+\mathbf{K}(k)\left(\mathbf{H}(k)\mathbf{P}(k|k-1)\mathbf{H}^T(k)+\mathbf{R}(k)\right)\mathbf{K}^T(k)\\ &=\mathbf{P}(k|k-1)-\mathbf{K}(k)\mathbf{H}(k)\mathbf{P}(k|k-1)-\mathbf{P}(k|k-1)\mathbf{H}^T(k)\mathbf{K}^T(k)+\mathbf{K}(k)\mathbf{S}(k)\mathbf{K}^T(k)\end{aligned}$$

The minimum mean-square error corresponds to trace of matrix $\mathbf{P}(k \mid k)$. The trace is minimized when the matrix derivative is zero

$$\frac{\partial\,\mathrm{tr}\left(\mathbf{P}(k|k)\right)}{\partial\mathbf{K}(k)}=-2\left(\mathbf{H}(k)\mathbf{P}(k|k-1)\right)^T+2\mathbf{K}(k)\mathbf{S}(k)=0$$

Solution of the previous matrix equation results in the formula for the Kalman gain

$$\mathbf{K}(k)\mathbf{S}(k)=\left(\mathbf{H}(k)\mathbf{P}(k|k-1)\right)^T=\mathbf{P}(k|k-1)\mathbf{H}^T(k)$$

$$\mathbf{K}(k)=\mathbf{P}(k|k-1)\mathbf{H}^T(k)\mathbf{S}^{-1}(k)$$

$$\frac{\partial\,\mathrm{tr}(\mathbf{AC})}{\partial\mathbf{A}}=\mathbf{C}^T$$

Note for **A C** to be square, dim **A** = dim $\mathbf{C}^T$

$$\frac{\partial\,\mathrm{tr}\left(\mathbf{ABA}^T\right)}{\partial\mathbf{A}}=2\mathbf{AB}$$

(where **B** is symmetric)

http://en.wikipedia.org/wiki/Kalman_filter

Simplification of the a posteriori error covariance formula

Multiplying both sides of the Kalman gain formula

$$\mathbf{K}(k)\mathbf{S}(k) = \mathbf{P}(k|k-1)\mathbf{H}^T(k)$$

on the right by $\mathbf{K}^T(k)$, we get

$$\mathbf{K}(k)\mathbf{S}(k)\mathbf{K}^T(k) = \mathbf{P}(k|k-1)\mathbf{H}^T(k)\mathbf{K}^T(k)$$

When analyzing the expanded formula for the *a posteriori* error covariance

$$\mathbf{P}(k|k) = \mathbf{P}(k|k-1) - \mathbf{K}(k)\mathbf{H}(k)\mathbf{P}(k|k-1) - \mathbf{P}(k|k-1)\mathbf{H}^T(k)\mathbf{K}^T(k) + \mathbf{K}(k)\mathbf{S}(k)\mathbf{K}^T(k)$$

we find that the last two terms cancel out

$$\mathbf{P}(k|k) = \mathbf{P}(k|k-1) - \mathbf{K}(k)\mathbf{H}(k)\mathbf{P}(k|k-1) = (\mathbf{E} - \mathbf{K}(k)\mathbf{H}(k))\mathbf{P}(k|k-1)$$

This formula is computationally cheaper and thus nearly always used in practice, but is only correct for the optimal gain. If arithmetic precision is unusually low causing problems with numerical stability, or if a non-optimal Kalman gain is deliberately used, this simplification cannot be applied; the a posteriori error covariance formula as derived above must be used.

http://en.wikipedia.org/wiki/Kalman_filter

Kalman filter - Algorithm

Initial estimate for

$\hat{\mathbf{x}}(-1|-1) \quad \mathbf{P}(-1|-1)$

Predict phase (starting phase for $k = 0$)

1) Project the state ahead

$$\hat{\mathbf{x}}(k|k-1) = \mathbf{A}(k)\hat{\mathbf{x}}(k-1|k-1) + \mathbf{B}(k)\mathbf{u}(k)$$

2) Project the error covariance ahead

$$\mathbf{P}(k|k-1) = \mathbf{A}(k)\mathbf{P}(k-1|k-1)\mathbf{A}^T(k) + \mathbf{Q}(k)$$

Measurement Update phase (Correct)

1) Compute the Kalman gain

$$\mathbf{K}(k) = \mathbf{P}(k|k-1)\mathbf{H}^T(k)\left(\mathbf{H}(k)\mathbf{P}(k|k-1)\mathbf{H}^T(k) + \mathbf{R}(k)\right)^{-1}$$

2) Update estimate with measurement $\mathbf{y}(k)$

$$\hat{\mathbf{x}}(k|k) = \hat{\mathbf{x}}(k|k-1) + \mathbf{K}(k)\left(\mathbf{y}(k) - \mathbf{H}(k)\hat{\mathbf{x}}(k|k-1)\right)$$

3) Update the error covariance

$$\mathbf{P}(k|k) = \left(\mathbf{E} - \mathbf{K}(k)\mathbf{H}(k)\right)\mathbf{P}(k|k-1)$$

$$k+1 \rightarrow k$$

http://en.wikipedia.org/wiki/Kalman_filter

Estimating a random constant with the use of Kalman filter

We assume that the measurement process is governed by difference equation

$$x(k+1) = x(k) + v_1(k) \qquad \text{a random walk}$$

$$y(k) = x(k) + v_2(k)$$

Algorithm

Initial values for process noise covariance Q and measurement noise covariance R

Initial estimate for $\hat{x}(-1|-1)$ and $P(-1|-1)$

$0 \rightarrow k$

$$\hat{x}(k|k-1) = \hat{x}(k-1|k-1) \qquad \text{; predict}$$

$$P(k|k-1) = P(k-1|k-1) + Q$$

$$K(k) = P(k|k-1)\left(P(k|k-1) + R\right)^{-1} \qquad \text{; update}$$

Input $y(k)$

$$\hat{x}(k|k) = \hat{x}(k|k-1) + K(k)\left(y(k) - \hat{x}(k|k-1)\right)$$

$$P(k|k) = (1 - K(k))P(k|k-1)$$

Output $\hat{x}(k|k)$, $K(k)$, $P(k|k)$

$k+1 \rightarrow k$

See [Welch & Bishop]

Let's assume that from experience we know that the true value of the random constant has a standard normal probability distribution, so we will "seed" our filter with the guess that the constant is zero.

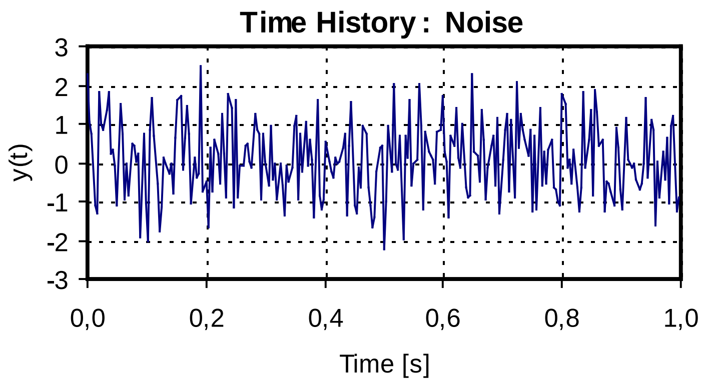
Similarly we need to choose an initial value for x. If we were absolutely certain that our initial state estimate x was correct, we would let $P(-1|-1) = 0$. However given the uncertainty in our initial estimate for x, choosing would cause the filter to always believe this value. As it turns out, the alternative choice is not critical. We could choose almost any and the filter would eventually converge.

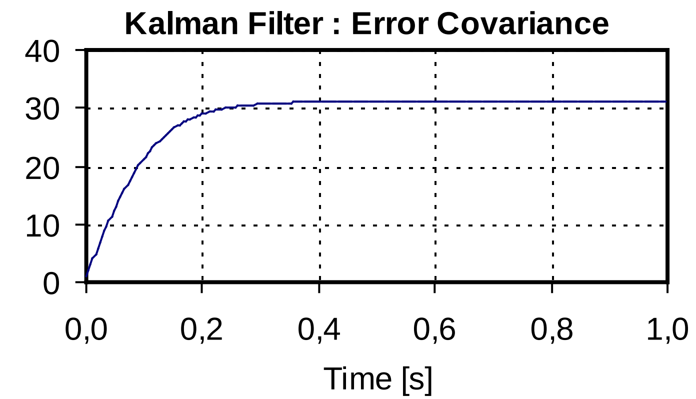
Starting the Kalman filter

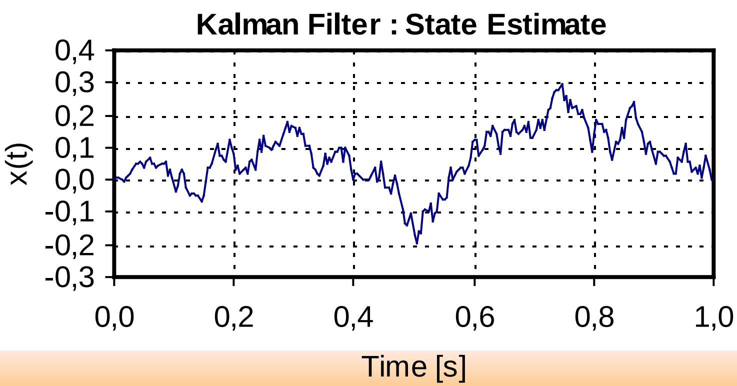
Let a random constant (zero) be measured repeatedly in a time sequence with an error of the unity covariance. We assume that the measurement process is governed by equation

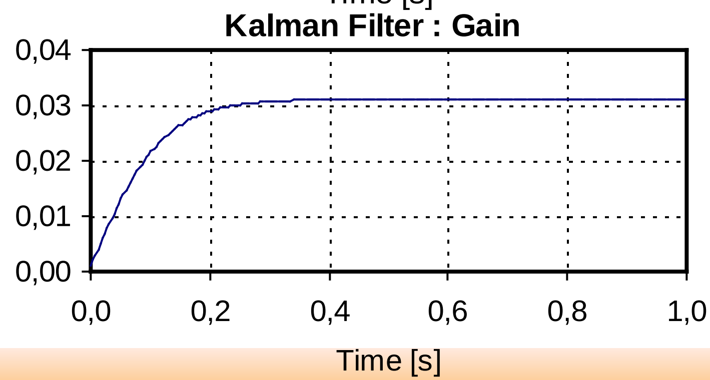
$$x(k+1) = x(k),\ \ y(k) = x(k) + v_2(k)$$

The covariance of the measurement noise $v_2(k)$ is equal to 1 while covariance of $v_2(k)$ is equal to 0. The initial values are as follows: $Q = 1$, $R = 1000$, initial guess of the state is 0 and initial guess of a posteriori error covariance is 1 as well



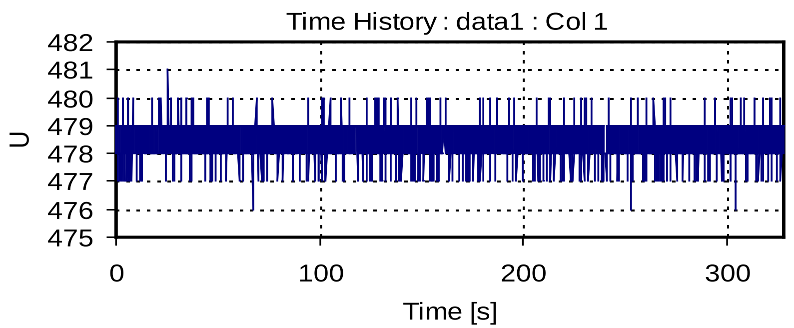




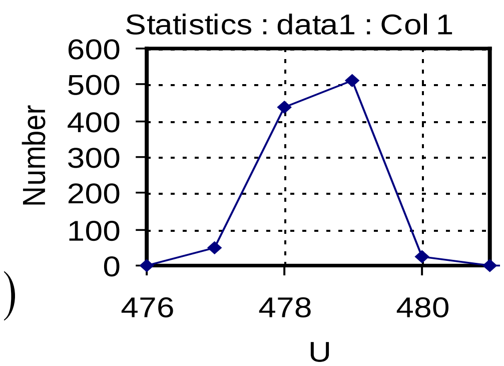


Example for estimating a random constant

ADC wideband noise due to the thermal effect

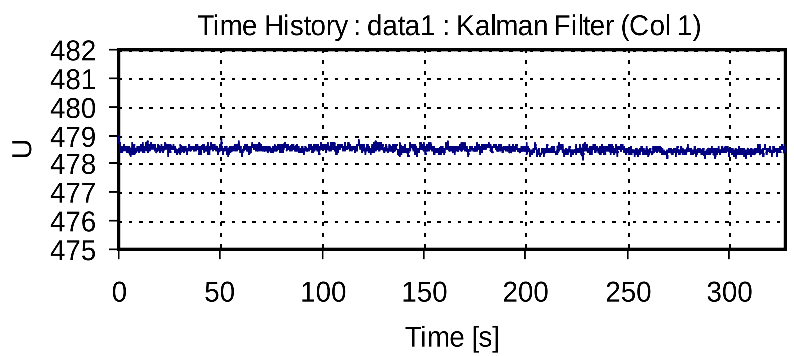


Grounded-input histogram



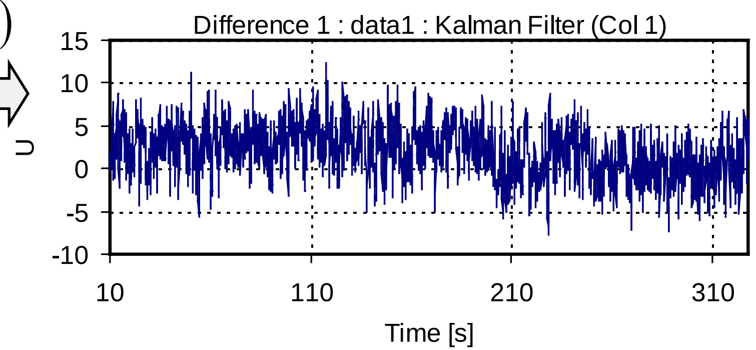
$y(k)$

Filter output



$x(k)$

Cumulative difference $\sum_{i=0}^{k}\left(y(i)-x(i)\right)$



Position and velocity of vehicle

Let the position of a vehicle be measured every Δt seconds, but these measurements are imprecise; we want to maintain a model of where the truck is and what its velocity is. The position and velocity of the truck is described by the linear state space

$$\mathbf{x}(k) = \begin{bmatrix} x(k) \\ \dot{x}(k) \end{bmatrix}$$

where x is the position and $\dot{x}$ is the velocity. It is assumed that between the $(k-1)$th and kth timestep the vehicle undergoes a constant acceleration of $a(k)$ that is normally distributed, $p(a(k)) \sim N\left(0, \sigma_a^2\right)$. From Newton's laws of motion it is possible to conclude that

$$\mathbf{x}(k) = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \mathbf{x}(k-1) + \begin{bmatrix} \Delta t^2/2 \\ \Delta t \end{bmatrix} a(k) \;\Rightarrow\; \mathbf{x}(k) = \mathbf{A}\,\mathbf{x}(k-1) + \mathbf{G}\,a(k) \;\Rightarrow\; \mathbf{x}(k) = \mathbf{A}\,\mathbf{x}(k-1) + \mathbf{v}_1(k)$$

where $p(\mathbf{v}_1(k)) \sim N(\mathbf{0}, \mathbf{Q})$

$$\mathbf{Q} = \mathbf{G}\mathbf{G}^T \sigma_a^2 = \begin{bmatrix} \Delta t^4/4 & \Delta t^3/2 \\ \Delta t^3/2 & \Delta t^2 \end{bmatrix} \sigma_a^2$$

At each time step, a noisy measurement of the true position $y(k)$ of the truck is made. Let us suppose the measurement noise $v_2(k)$ is also normally distributed, $p(v_2(k)) \sim N\left(0, \sigma_y^2\right)$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(k) + v_2(k) \quad\Rightarrow\quad y(k) = \mathbf{H}\,\mathbf{x}(k) + v_2(k)$$

where $R = E\left\{v_2(k) v_2^T(k)\right\} = \left[\sigma_y^2\right]$

The initial starting state of the vehicle with perfect precision $\hat{\mathbf{x}}(-1|-1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad \mathbf{P}(-1|-1) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Kalman–Bucy filter

The Kalman–Bucy filter is a continuous time version of the Kalman filter. A mathematical model is of the state space type

$$\frac{d}{dt}\mathbf{x}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{v}_1(t)$$

$$\mathbf{y}(t) = \mathbf{H}(t)\mathbf{x}(t) + \mathbf{v}_2(t)$$

where the covariances of the noise terms $\mathbf{v}_1(t)$ and $\mathbf{v}_2(t)$ are given by $\mathbf{Q}(t)$ and $\mathbf{R}(t)$, respectively.

The filter consists of two differential equations, one for the state estimate and one for the covariance

$$\frac{d}{dt}\hat{\mathbf{x}}(t) = \mathbf{A}(t)\hat{\mathbf{x}}(t) + \mathbf{K}(t)\left(\mathbf{y}(t) - \mathbf{H}(t)\hat{\mathbf{x}}(t)\right)$$

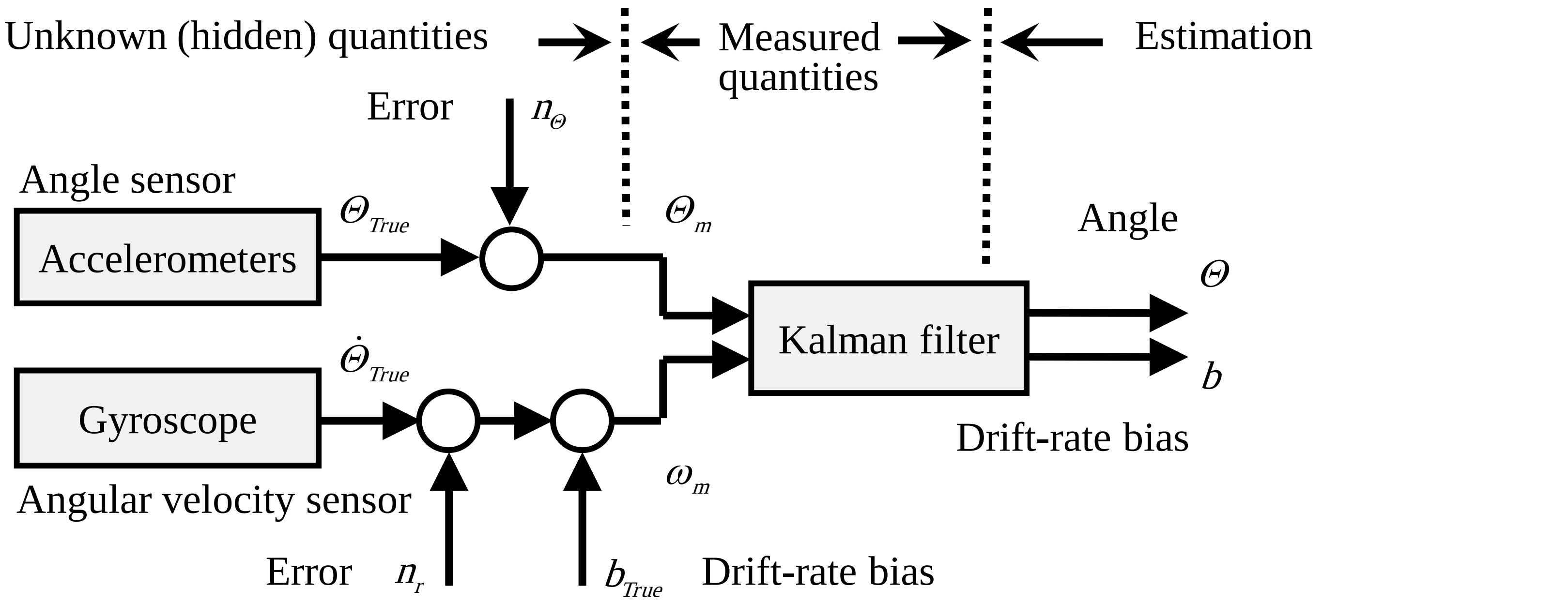
$$\frac{d}{dt}\mathbf{P}(t) = \mathbf{A}(t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{A}^T(t) + \mathbf{Q}(t) - \mathbf{K}(t)\mathbf{R}(t)\mathbf{K}^T(t)$$

where the Kalman gain is given by

$$\mathbf{K}(t) = \mathbf{P}(t)\mathbf{H}^T(t)\mathbf{R}^{-1}(t)$$

Example - Inertial navigation system (1 DOF)

The continuous time Kalman filter used for planar motion with a single bias compensated gyro



Continuous time t, angle Θ, angular velocity ω, drift-rate bias b

Measured quantities Θ_m, ω_m

Unknown quantities Θ_{True}, ω_{True}

Gauss white-noise error covariances $E\{(n_r(t))^2\} = N_r,\; E\{(n_\Theta(t))^2\} = N_\Theta$

See [Roumeliotis & Sukhatme & Bekey]

Kalman Filter – Process equation

A mathematical model and a matrix form corresponding to the block schema on the previous slide

$$\dot{\Theta}_{True} = \omega_m + b_{True} + n_r \qquad \Rightarrow \qquad \frac{d}{dt}\begin{bmatrix} \Theta_{True} \\ b_{True} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \Theta_{True} \\ b_{True} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\omega_m + \begin{bmatrix} n_r \\ n_w \end{bmatrix} \qquad E\left\{(n_w(t))^2\right\} = N_w$$
$$\dot{b}_{True} = n_w$$

The term ω_m is like a control input u to the system and needs to be eliminated. This can be done either to add it to the state and estimate it or to formulate the estimation algorithm as an Indirect Kalman filter since the orientation error is estimated instead of directly estimating orientation. The orientation estimate obtained by integrating the gyro signal (assuming constant bias b_i) is given by

$$\dot{\Theta}_i = \omega_m + b_i \qquad \Rightarrow \qquad \frac{d}{dt}\begin{bmatrix} \Theta_i \\ b_i \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \Theta_i \\ b_i \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\omega_m$$
$$\dot{b}_i = 0$$

Subtracting both the models we obtain $\Delta\dot{\Theta} = b_i + n_r$

The variable $\Delta\Theta$ is the error in orientation and Δb is the bias error. Subtracting the equations for b_{True} and b_i the bias error can be written as $\dot{b} = n_w$. These error propagation equations for the Indirect (error state) Kalman filter can be rearranged as

$$\frac{d}{dt}\begin{bmatrix} \Delta\Theta \\ \Delta b \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \Delta\Theta \\ \Delta b \end{bmatrix} + \begin{bmatrix} n_r \\ n_w \end{bmatrix}$$

or in a more compact form as …….. $\Delta\mathbf{x} = \begin{bmatrix} \Delta\Theta \\ \Delta b \end{bmatrix} \quad \Rightarrow \quad \dfrac{d\Delta\mathbf{x}}{dt} = \mathbf{A}\,\Delta\mathbf{x} + \mathbf{n}$

Kalman Filter – Measurement equation

We assume that the measurement provided to the Indirect Kalman filter is

$$\Delta y = \Theta_m - \Theta_i = \Theta_{True} + n_\Theta - \Theta_i = \Delta\Theta + n_\Theta$$

where Θ_i is available through the gyro signal integration and Θ_m , is the absolute orientation measurement. This equation in matrix form becomes

$$\Delta y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta\Theta \\ \Delta b \end{bmatrix} + n_\Theta \qquad \Rightarrow \qquad \Delta y = \mathbf{H}\ \Delta\mathbf{x} + n_\Theta$$

The continuous Kalman filter equation for the covariance **P** is

$$\dot{\mathbf{P}} = \mathbf{AP} + \mathbf{PA}^T + \mathbf{Q} - \mathbf{PH}^T\mathbf{R}^{-1}\mathbf{HP} \qquad \text{where} \qquad \mathbf{Q} = \begin{bmatrix} N_r & 0 \\ 0 & N_w \end{bmatrix}, \ \mathbf{R} = \begin{bmatrix} N_\Theta \end{bmatrix}$$

The Kalman gain at the steady-state operation

$$\dot{\mathbf{P}} = 0 \quad \Rightarrow \quad \mathbf{K} = \mathbf{PH}^T\mathbf{R}^{-1} = \begin{bmatrix} \sqrt{\left(N_r + \sqrt{N_w N_\Theta}\right)/N_\Theta} \\ \sqrt{N_w/N_\Theta} \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

The estimate propagation equation with the added correction is as follows

$$\frac{d}{dt}\begin{bmatrix} \Delta\hat{\Theta} \\ \Delta\hat{b} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\hat{\Theta} \\ \Delta\hat{b} \end{bmatrix} + \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \left(\Delta y - \Delta\hat{\Theta}\right)$$

Kalman Filter – Measurement equation - cont'd

Substituting the error state estimates

$$\Delta\hat{\Theta} = \hat{\Theta} - \Theta_i, \quad \Delta\hat{b} = \hat{b} - b_i$$

we have

$$\frac{d}{dt}\begin{bmatrix} \hat{\Theta} - \Theta_i \\ \hat{b} - b_i \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \hat{\Theta} - \Theta_i \\ \hat{b} - b_i \end{bmatrix} + \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}\left(\Delta y - \Delta\hat{\Theta}\right)$$

Separating the estimated and integrated quantities results in

$$\frac{d}{dt}\begin{bmatrix} \hat{\Theta} \\ \hat{b} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \hat{\Theta} \\ \hat{b} \end{bmatrix} + \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}\left(\Delta y - \Delta\hat{\Theta}\right) + \left(\frac{d}{dt}\begin{bmatrix} \Theta_i \\ b_i \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \Theta_i \\ b_i \end{bmatrix}\right)$$

Notice that

$$\Delta y - \Delta\hat{\Theta} = \left(\Theta_m - \Theta_i\right) - \left(\hat{\Theta} - \Theta_i\right) = \Theta_m - \hat{\Theta}$$

After substitution of the term resulting from integration we get

$$\frac{d}{dt}\begin{bmatrix} \hat{\Theta} \\ \hat{b} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \hat{\Theta} \\ \hat{b} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\omega_m + \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}\left(\Theta_m - \hat{\Theta}\right)$$

The Laplace transform of the angle estimate results in

$$\hat{\Theta}(s) = \frac{s^2}{s^2 + k_1 s + k_2}\frac{\Omega_m(s)}{s} + \frac{k_1 s + k_2}{s^2 + k_1 s + k_2}\Theta_m(s), \quad F(s) = \frac{s^2}{s^2 + k_1 s + k_2}$$

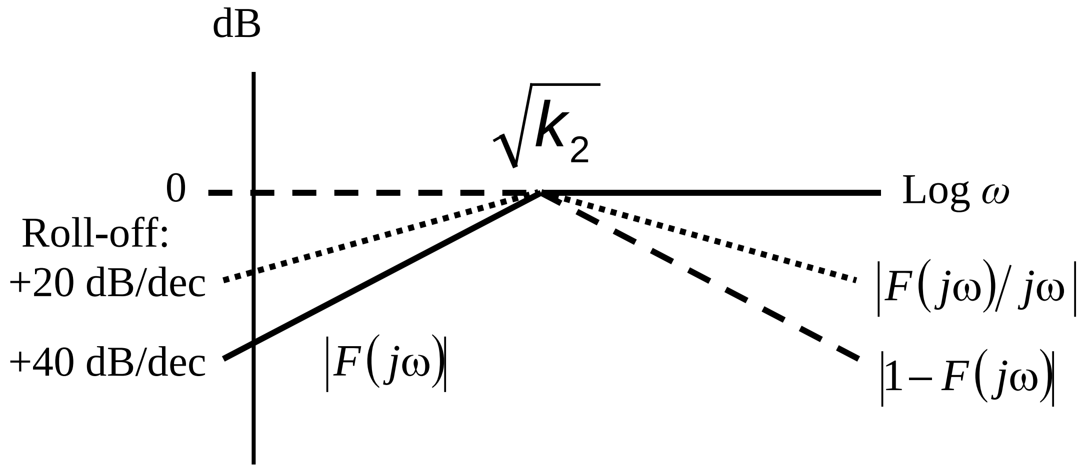
Signal Fusion - Frequency Response Function

The previous formula can be rewritten as

$$\hat{\Theta}(s) = F(s)\frac{\Omega_m(s)}{s} + (1 - F(s))\,\Theta_m(s)$$

integration

After replacing $s = j\omega$ we get a frequency response



See [Roumeliotis & Sukhatme & Bekey]

Extended Kalman Filter

Let us assume that the process again has a state vector **x**, but that the process is now governed by the non-linear stochastic difference equation

$$\mathbf{x}(k) = f\left(\mathbf{x}(k-1), \mathbf{u}(k), \mathbf{v}_1(k)\right)$$

with the measurement (observation) vector **y** that is defined by a measurement equation, describing the observation as

$$\mathbf{y}(k) = h\left(\mathbf{x}(k), \mathbf{v}_2(k)\right)$$

where random variables $\mathbf{v}_1$ a $\mathbf{v}_2$ represent the process and measurement noise (respectively). It is assumed that the random variables are independent of each other and with normal probability distribution.

In practice of course one does not know the individual values of the noise and at each time step. However, one can approximate the state and measurement vector without them as

$$\tilde{\mathbf{x}}(k) = f\left(\tilde{\mathbf{x}}(k-1), \mathbf{u}(k), 0\right)$$

$$\tilde{\mathbf{y}}(k) = h\left(\tilde{\mathbf{x}}(k), 0\right)$$

After linearization we get

$$\mathbf{x}(k) \approx \hat{\mathbf{x}}(k) + \mathbf{A}\left(\mathbf{x}(k-1) - \hat{\mathbf{x}}(k-1)\right) + \mathbf{W}\mathbf{v}_1(k)$$

$$\mathbf{y}(k) \approx \hat{\mathbf{y}}(k) + \mathbf{H}\left(\mathbf{x}(k) - \hat{\mathbf{x}}(k)\right) + \mathbf{V}\mathbf{v}_2(k)$$

where $A_{i,j} = \dfrac{\partial f_i\left(\hat{x}(k), u(k), 0\right)}{\partial x_j}, W_{i,j} = \dfrac{\partial f_i\left(\hat{x}(k), u(k), 0\right)}{\partial v_{1,j}}, H_{i,j} = \dfrac{\partial h_i\left(\hat{x}(k), 0\right)}{\partial x_j}, H_{i,j} = \dfrac{\partial h_i\left(\hat{x}(k), 0\right)}{\partial v_{2,j}}$

See [Welch & Bishop]

Extended Kalman filter - Algorithm

Initial estimate for

$\hat{\mathbf{x}}(-1|-1) \quad \mathbf{P}(-1|-1)$

Predict phase (starting phase for k = 0)

1) Project the state ahead

$$\hat{\mathbf{x}}(k|k-1) = f\left(\hat{\mathbf{x}}(k-1|k-1), \mathbf{u}(k)\right)$$

2) Project the error covariance ahead

$$\mathbf{P}(k|k-1) = \mathbf{A}(k)\mathbf{P}(k-1|k-1)\mathbf{A}^T(k) + \\ + \mathbf{W}(k)\mathbf{Q}(k)\mathbf{W}^T(k)$$

Measurement Update phase (Correct)

1) Compute the Kalman gain

$$\mathbf{K}(k) = \mathbf{P}(k|k-1)\mathbf{H}^T(k) \\ \left(\mathbf{H}(k)\mathbf{P}(k|k-1)\mathbf{H}^T(k) + \mathbf{V}(k)\mathbf{R}(k)\mathbf{V}^T(k)\right)^{-1}$$

2) Update estimate with measurement $\mathbf{y}(k)$

$$\hat{\mathbf{x}}(k|k) = \hat{\mathbf{x}}(k|k-1) + \\ + \mathbf{K}(k)\left(\mathbf{y}(k) - h\left(\hat{\mathbf{x}}(k|k-1)\right), 0\right)$$

3) Update the error covariance

$$\mathbf{P}(k|k) = \left(\mathbf{E} - \mathbf{K}(k)\mathbf{H}(k)\right)\mathbf{P}(k|k-1)$$

$$k+1 \rightarrow k$$

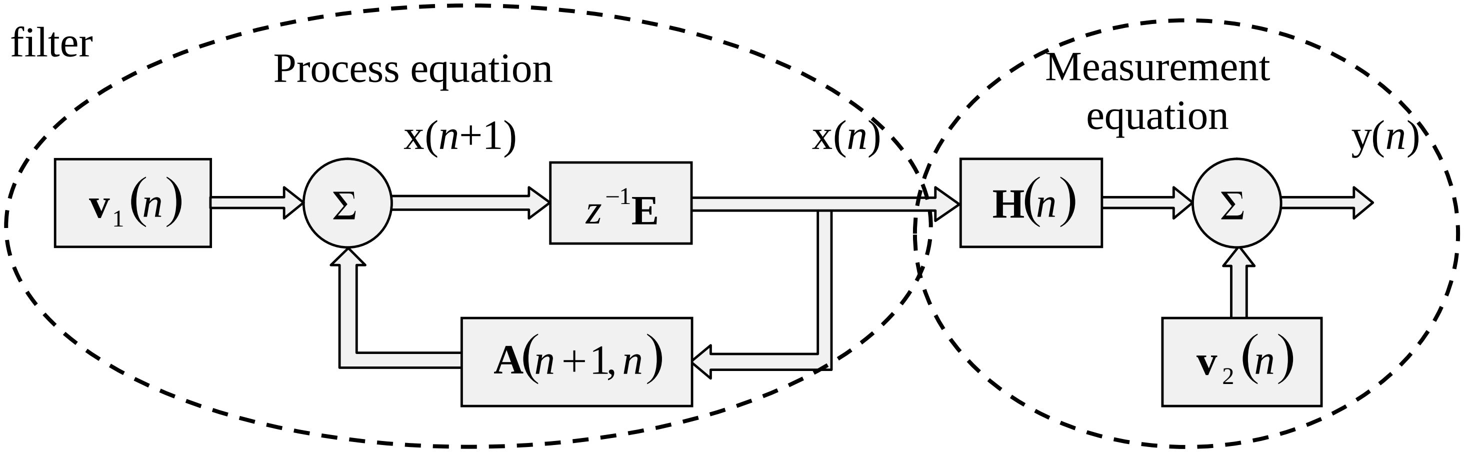
http://en.wikipedia.org/wiki/Kalman_filter

Håvard Vold, Ph.D. *1947

VOLD-KALMAN ORDER TRACKING FILTER

Kalman filter vs. Vold-Kalman filter

The Kalman filter



where

$\mathbf{v}_1(n)$ is uncorrelated excitation vector of process equation

$\mathbf{v}_2(n)$ is uncorrelated excitation vector of measurement equation

Input parameters: matrix **A** defining a process equation, matrix **H** defining measurement equation, covariance matrices $\mathbf{v}_1(n)$ and $\mathbf{v}_2(n)$

The Vold-Kalman filter

Input parameters: structural equation as an equivalent of the process equation,

data equation as an equivalent of the measurement equation, and

relationship between the norm of both the excitation vectors.

Software for the Vold-Kalman order filtration

The Vold-Kalman order filter is a bandpass filter, the center frequency of which can be continuously changed according to the instantaneous rotational frequency of a machine. The Vold-Kalman filter tracks the spectrum components of the input signal, called as orders, the frequency of which are multiples of the mentioned rotational frequency. It is assumed that the rotational frequency is defined for each sample of the input signal . Because the rotational frequency is usually measured with the use of a tacho signal producing the average rotational speed during a time interval, for estimating the instantaneous value of RPM the cubic spline curve fitting method has to be used for example.

The Vold-Kalman filter was developed in two generations. The output of the first generation is the filtered signal while the output of the second one is the envelope of the filtered signal.

The second generation only

- Brüel & Kjær, LabShop PULSE, Software Type 7703
- MTS Systems Corporation, I-DEAS

The first and second generation

- VSB – Technical University of Ostrava
  - M-functions in MATLAB including crossing orders (open code)
  - Signal Analyzer, indor software (VB6 - without crossing orders)
- Axiom-EduTech Sweden & VSB – TU Ostrava, M-functions in MATLAB (open code)

See [Vold & Leuridan]

Data equation (eq. to Kalman's measurement equation)

The data equation decomposes a signal $y(n)$, where $n = 1,\dots, N$, into two parts: the filter output and an error term $\eta(n)$.
Data equations for extraction of one component ($P = 1$), which is modeled by a structural equation

The first generation

$$y(n) = x(n) + \eta(n)$$

$x(n)$ – filter output as a real signal

The second generation T_S – sampling interval

$$y(n) = x(n)\exp(j\Theta(n)) + \eta(n) \qquad \Theta(n) = \sum_{i=0}^{n} \omega(i) T_S$$

$\Theta(n)$ – signal phase, $\omega(n)$ – angular frequency
$x(n)$ – complex envelope as the filter output

Data equations for extraction of P signal components, each of them is modeled by individual structural equation

$$y(n) = \sum_{i=1}^{P} x_i(n) + \eta(n) \qquad \Big| \qquad y(n) = \sum_{i=1}^{P} x_i(n)\exp(j\Theta_i(n)) + \eta(n)$$

The vector form of the input and output data

$$\mathbf{y} = [y(1),\dots, y(N)]^T,\ \boldsymbol{\eta} = [\eta(1),\dots,\eta(N)]^T,\ \mathbf{x}_i = [x_i(1),\dots, x_i(N)]^T,\ \mathbf{C}_i = \mathrm{diag}\{\exp(j\Theta_i(1)),\dots,\exp(j\Theta_i(N))\}$$

The matrix form of data equations

$$\Theta_i(n) = \sum_{i=0}^{n} \omega_i(i) T_S$$

$$\mathbf{y} - (\mathbf{x}_1 + \dots + \mathbf{x}_P) = \boldsymbol{\eta} \qquad \Big| \qquad \mathbf{y} - (\mathbf{C}_1\mathbf{x}_1 + \dots + \mathbf{C}_P\mathbf{x}_P) = \boldsymbol{\eta}$$

To asses difference between $y(n)$ and $x_i(n)$, the square of the error vector norm is introduced

$$\boldsymbol{\eta}^T\boldsymbol{\eta} = (\mathbf{y}^T - \mathbf{x}_1^T - \dots - \mathbf{x}_P^T)(\mathbf{y} - \mathbf{x}_1 - \dots - \mathbf{x}_P) \qquad \Big| \qquad \boldsymbol{\eta}^H\boldsymbol{\eta} = (\mathbf{y}^T - \mathbf{x}_1^T\mathbf{C}_1^H - \dots - \mathbf{x}_P^T\mathbf{C}_P^H)(\mathbf{y} - \mathbf{C}_1\mathbf{x}_1 - \dots - \mathbf{C}_P\mathbf{x}_P)$$

See [Vold & Leuridan]

The matrix **C** is the unit matrix for the first generation filtr.

Solution of the homogenous difference equations

The structural equation is a generator of a signal

The first generation of the Vold-Kalman order filter

$$x(n) - c(n)x(n-1) + x(n-2) = 0$$

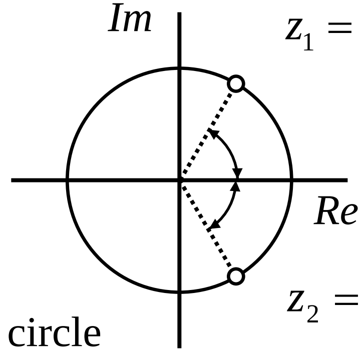
ω – rotational speed (interpolated),
T_S – sampling interval

The solution of the homogeneous difference equation

$$x(n) = az_1^n + bz_2^n, \quad c(n) = 2\cos(\omega T_S)$$

$z_2 = z_1^*$ complex conjugate roots of a characteristic equation

$$x(n) = A_\omega \cos(\omega\, nT_S + \varphi)$$



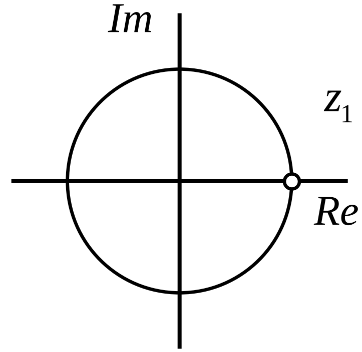
Approximation by a harmonic function

The second generation for the one-pole filter

$$x(n) - x(n-1) = 0 \quad \Rightarrow \quad z_1 = 1$$

$$x(n) = z_1^n \quad \Rightarrow \quad x(n) = 1$$

Piecewise approximation by a constant



The second generation for the two-pole filter

$$x(n) - 2x(n-1) + x(n-2) = 0 \quad \Rightarrow \quad z_1 = 1 \quad \text{double root}$$

$$x(n) = az_1^n + bnz_1^n \quad \Rightarrow \quad x(n) = a + cnT_S, \quad (cT_S = b)$$

Piecewise approximation by a strait line

Structural equation (eq. to Kalman's process equation)

To simplify formulas the index of the signal component is omited.

The first generation of the Vold-Kalman order filter

$$x(n) - 2\cos(\omega T_S)x(n-1) + x(n-2) = \varepsilon(n)$$

$$c(n) = 2\cos(\omega T_S)$$

ω – rotational speed (interpolated),
$x(n)$ – filtered signal, $\varepsilon(n)$ – error term, N – sample number, T_S – sampling interval

The second generation

$x(n) - x(n-1) = \varepsilon(n)$ … one-pole filter

$x(n) - 2x(n-1) + x(n-2) = \varepsilon(n)$ … two-pole filter

$x(n) - 3x(n-1) + 3x(n-2) - x(n-3) = \varepsilon(n)$ … three-pole filter

$x(n) - 4x(n-1) + 6x(n-2) - 4x(n-3) + x(n-4) = \varepsilon(n)$ … four-pole filter

The matrix form of the structural equation for the *i*-th component is as follows

$$\mathbf{A}_i \, \mathbf{x}_i = \boldsymbol{\varepsilon}_i$$

To asses the error term $\varepsilon(n)$, as an exciting function for the structural function, the sum of the error term square (the square of the vector norm) is introduced

$\boldsymbol{\varepsilon}_i^T \boldsymbol{\varepsilon}_i = \mathbf{x}_i^T \mathbf{A}_i^T \mathbf{A}_i \, \mathbf{x}_i$ $\qquad$ $\mathbf{A}_i = \mathbf{A}$ … for the second generation filter

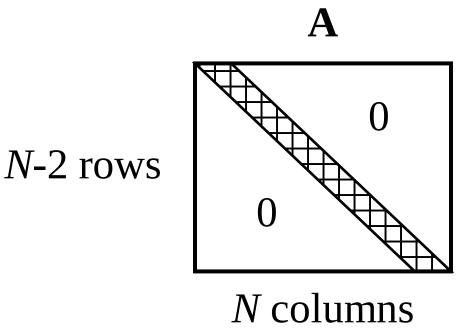
Matrix forms of the structural equations

The first generation of the Vold-Kalman order filter

$$n = 3, \ldots N: \quad x(n) - c(n)\,x(n-1) + x(n-2) = \varepsilon(n)$$

$$\begin{bmatrix} 1 & -c & 1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 1 & -c & 1 & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & 1 & -c & 1 \end{bmatrix} \begin{bmatrix} x(1) \\ x(2) \\ \ldots \\ x(N) \end{bmatrix} = \begin{bmatrix} \varepsilon(3) \\ \varepsilon(4) \\ \ldots \\ \varepsilon(N) \end{bmatrix}$$

Sparse band matrix



The second generation, the example for the two-pole filter

$$n = 3, \ldots, N: \quad x(n) - 2\,x(n-1) + x(n-2) = \varepsilon(n)$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} x(1) \\ x(2) \\ \ldots \\ x(N) \end{bmatrix} = \begin{bmatrix} \varepsilon(3) \\ \varepsilon(4) \\ \ldots \\ \varepsilon(N) \end{bmatrix}$$

$$\mathbf{A}\,\mathbf{x} = \boldsymbol{\varepsilon}$$

Global solution

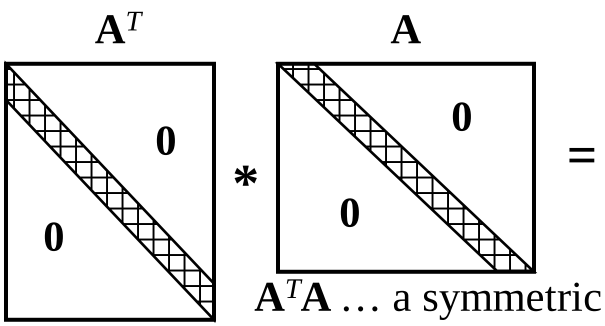
The unknown vector **x** is composed from *N* samples. The count of the structural and data equations is greater than the count of unknown quantities. The system of the linear equations is over-determined. To find a solution of the equations, an objective function is added to reach the required relationship between influence of the data and structural equations on the result. Minimizing the objective function can be done by putting the first derivative with respect to the unknown vector to zero.

The objective function is as follows

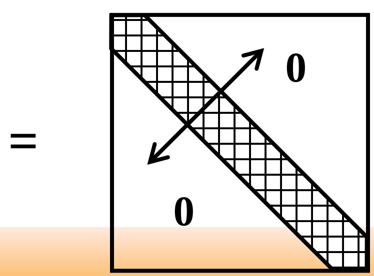
$$J = r^2 \boldsymbol{\varepsilon}^T \boldsymbol{\varepsilon} + \boldsymbol{\eta}^T \boldsymbol{\eta} \rightarrow \min$$

where

r – weighting coefficient



$\mathbf{A}^T\mathbf{A}$ … a symmetric matrix



The solution is as follows

The first generation filter

$$\frac{\partial J}{\partial \mathbf{x}} = 2 r^2 \mathbf{A}^T \mathbf{A}\,\mathbf{x} + 2(\mathbf{x} - \mathbf{y}) = \mathbf{0}$$

$$\mathbf{x} = \left(r^2 \mathbf{A}^T \mathbf{A} + \mathbf{E}\right)^{-1} \mathbf{y}$$

The second generation filter

$$\frac{\partial J}{\partial \mathbf{x}^H} = \left(r^2 \mathbf{A}^T \mathbf{A} + \mathbf{E}\right)\mathbf{x} - \mathbf{C}^H \mathbf{y} = \mathbf{0}$$

$$\mathbf{x} = \left(r^2 \mathbf{A}^T \mathbf{A} + \mathbf{E}\right)^{-1} \mathbf{C}^H \mathbf{y} \qquad \mathbf{B} = r^2\mathbf{A}^T\mathbf{A}+\mathbf{E}$$

Adding unity to the matrix main diagonal turns a symmetric matrix to the Symmetric Positive Definite matrix (SPD)

See [Tůma,2005]

Robustness of the linear equation solution

1) The high selectivity of the Vold-Kalman filter requires to assign a value of the weighting coefficient r to hundreds or even thousands

2) Elements of $\mathbf{A}^T\mathbf{A}$ are as follows

The first generation

$$\mathbf{A}^T\mathbf{A} = \begin{pmatrix} 1 & -c_1 & 1 & 0 & \dots \\ -c_1 & c_1^2+1 & -c_1-c_2 & 1 & \dots \\ 1 & -c_1-c_2 & c_2^2+2 & -c_2-c_3 & \dots \\ 0 & 1 & -c_2-c_3 & c_3^2+2 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

$$|c_i| \leq 2$$

The second generation (3-pole filter)

$$\mathbf{A}^T\mathbf{A} = \begin{pmatrix} 1 & -3 & 3 & -1 & & \\ -3 & 10 & -12 & 6 & -1 & \\ 3 & -12 & 19 & -15 & 6 & \dots \\ -1 & 6 & -15 & 20 & -15 & \dots \\ & -1 & 6 & -15 & 20 & \dots. \\ & & \dots & \dots & \dots & \dots \end{pmatrix}$$

$r^2\mathbf{A}^T\mathbf{A}$ is generally only a positive semidefinite matrix, adding the unity matrix $\mathbf{E}$ turns it to the positive definite matrix

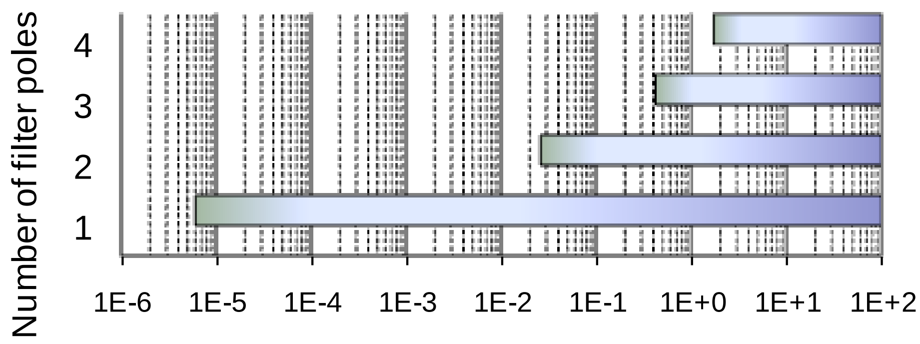
Limit values of the weighting coefficient

The second generation filter of the Vold-Kalman order filter

The value of the weighting coefficient r should be limited not to lost the effect of adding the unit to the main matrix diagonal by rounding due to the limit number of bits (double precision number is assumed) for saving quantities in a computer memory

Number of poles	$p = 1$	$p = 2$	$p = 3$	$p = 4$
$(r^2\mathbf{A}^T\mathbf{A})_{i,i}$	$2r^2$	$6r^2$	$20r^2$	$70r^2$
$r_{MAX} \approx$	$7\text{x}10^6$	$4\text{x}10^6$	$2\text{x}10^6$	$1.1\text{x}10^6$
$100\Delta f >$	$7\text{x}10^{-6}$ %	0.025 %	0.5 %	2 %

The number of poles is designated as p and the relative bandwidth is defined as $\Delta f = 2 f_H / f_S$

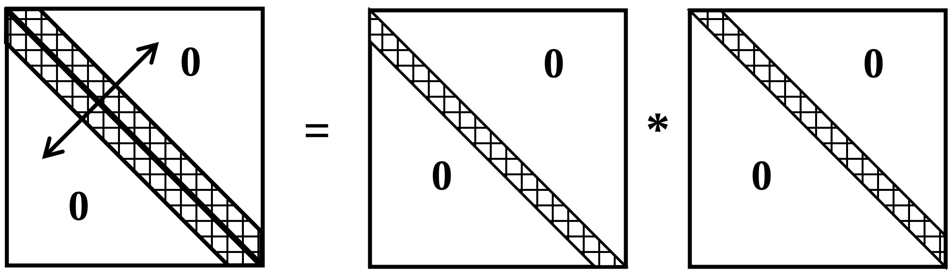


See [Tůma,2005]

Range of the pass-band filter bandwidth in %

Cholesky factorization for solution of the linear equation system

B = LU $\quad\quad$ **L = U**^{T} $\quad\quad$ **U** = $\{u_{i,j}\}$



The algorithm for the 1st generation filter

$$u_{1,2} = b_{1,2} / u_{1,1}$$
$$u_{1,1} = \sqrt{b_{1,1}}$$
$$u_{2,2} = \sqrt{b_{2,2} - u_{1,2}^2}$$
$$j = 3, \ldots, N$$
$$u_{j-2,j} = b_{j-2,j} / u_{j-2,j-2}$$
$$u_{j-1,j} = \left(b_{j-1,j} - u_{j-2,j-1} u_{j-2,j}\right) / u_{j-1,j-1}$$
$$u_{j,j} = \sqrt{b_{j,j} - u_{j-1,j}^2 - u_{j-2,j}^2}$$

L,**U** triangular matrices

LU x = y

Substitution **U x = z**

gives **L z = y**

z = L^{-1}**y** $\quad$ => $\quad$ **x = U**$^{-1}$**z**

Fot the first generation filter the matrices **L** and **U** have 3 non-zero diagonals

The Cholesky factorization of a positive definite matrix saves the band property of resulting triangular matrices

Solution of the linear equation system as a filtration process

Forward reduction $\mathbf{L}\,\mathbf{z} = \mathbf{y}$ $(\mathbf{L} = \mathbf{U}^T)$	Backward substitution $\mathbf{U}\,\mathbf{x} = \mathbf{z}$
$z_1 = y_1 / u_{1,1}$	$x_N = z_1 / u_{N,N}$
$z_2 = \left(y_2 - u_{1,2} z_1\right) / u_{2,2}$	$x_{N-2} = \left(z_{N-2} - u_{N-1,N} x_N\right) / u_{N-1,N-1}$
…..	…..
$j = p+1, \ldots, N$	$j = N-(p+1), \ldots, 1$ (reverse order)
$z_j = \left(y_j - u_{j-1,j}\, z_{j-1} \ldots - u_{j-p,j}\, z_{j-p}\right) / u_{j,j}$	$x_j = \left(z_j - u_{j,j+1} x_{j+1} \ldots - u_{j,j+p} x_{j+p}\right) / u_{j,j}$

Steady-state values …. $u_0 \rightarrow u_{j,j},\, u_1 \rightarrow u_{j,j+1}, \ldots\; u_p \rightarrow u_{j,j+p}$

The transfer functions of the *p*-order IIR filter in the Z-transform

$$H_F(z) = \frac{Z(z)}{Y(z)} = \frac{1}{u_0 + u_1 z^{-1} + \ldots + u_p z^{-p}} \qquad H_B(z) = \frac{X(z)}{Z(z)} = \frac{1}{u_0 + u_1 z + \ldots + u_p z^{p}}$$

The forward reduction and backward substitution results in the zero-phase digital filter, which is analogous to the *filtfilt* function in Matlab

$Y(z)$ → [Forward reduction $H_F(z)$] → $Z(z)$ → [Backward substitution $H_B(z)$] → $X(z)$

The first generation VK-filter, the steady-state values of the filter coefficients

The values of the matrix elements $\mathbf{B} = r^2 \mathbf{A}^T \mathbf{A} + \mathbf{E}$ are given by

$$b_0 = b_{j,j} = r^2\left(2 + c^2\right) + 1, \quad b_1 = b_{j,j+1} = b_{j,j-1} = -2cr^2, \quad b_2 = b_{j,j+2} = b_{j,j-2} = r^2$$

The Cholesky factorization of $\mathbf{B}$ results in $u_2 = b_2 / u_0$, $u_1 = \left(b_1 - u_1 u_2\right)/u_0$, $u_0 = \sqrt{b_0 - u_1^2 - u_2^2}$

$$u_0 u_2 = b_2, \quad u_0 u_1 + u_1 u_2 = b_1, \quad u_0^2 + u_1^2 + u_2^2 = b_0$$

By using the substitution $e^{-j\Omega} = \cos(\Omega) - j\sin(\Omega)$ it is obtained

$$\left|H\left(e^{j\Omega}\right)\right|^2 = \frac{1}{u_0^2 + u_1^2 + u_2^2 + 2\left(u_0 u_1 + u_1 u_2\right)\cos(\Omega) + 2u_0 u_2 \cos(2\Omega)}$$

The 3-dB bandwidth $\Delta f = 2\left(f_H - f_S\right)/ f_S$ of the filters $H_F(z)$ and $H_B(z)$, connected in series, may be obtained by substitutions $u_0 u_2 = b_2, u_0 u_1 + u_1 u_2 = b_1$ and $u_0^2 + u_1^2 + u_2^2 = b_0$

$$\left|H\left(e^{j\Omega}\right)\right|^2 = \frac{1}{b_0 + 2b_1\cos(\Omega) + 2b_2\cos(2\Omega)} = \frac{1}{\sqrt{2}}$$

Solution of the previous equation gives (T_S – the sampling interval, ω_C – the filter central frequency)

$$\Delta f = \frac{1}{\pi}\left(\arccos\left(\cos\left(\omega T_S\right) - \sqrt{\sqrt{2}-1}/2r\right) - \arccos\left(\cos\left(\omega T_S\right) + \sqrt{\sqrt{2}-1}/2r\right)\right)$$

Assuming $\Omega = \omega_C T_S + \Delta\Omega, \cos(\Delta\Omega) \approx 1, \sin(\Delta\Omega) \approx \Delta\Omega$ we obtain $r \approx \dfrac{1}{\pi \Delta f}\dfrac{\sqrt{\sqrt{2}-1}}{\sqrt{1 - \left(\cos\left(\omega_C T_S\right)\right)^2}}$

The second generation VK-filter, the steady-state values of the filter coefficients for the one-pole filter

The values of the matrix elements $\mathbf{B} = r^2 \mathbf{A}^T \mathbf{A} + \mathbf{E}$ are given by

$$b_0 = b_{j,j} = 2r^2 + 1, \quad b_1 = b_{j,j+1} = b_{j,j-1} = -r^2$$

The Cholesky factorization of the matrix **B** results in

$$u_{j-1,j} = b_{j-1,j} / u_{j-1,j-1}, \quad u_{j,j} = \sqrt{b_{j,j} - u_{j-1,j}^2}$$

For the steady-state values of the filter coefficients we obtain

$$u_1 = b_1 / u_0, \quad u_0 = \sqrt{b_0 - u_1^2} \qquad \Rightarrow \qquad u_0 u_1 = b_1, \quad u_0^2 + u_1^2 = b_0$$

The 3-dB bandwidth of two one-pole low pass in series filter results from the value of the frequency response

$$\left| G\left(e^{j\Omega}\right)_{LP} \right|^2 = \left| \frac{1}{u_0 + u_1 e^{-j\Omega}} \right|^2 = \frac{1}{\sqrt{2}}$$

Solution gives a value of the weighting coefficient r as a function of the relative bandwidth

$$r = \sqrt{\frac{\sqrt{2} - 1}{2\left(1 - \cos\left(\pi \Delta f\right)\right)}}$$

The relative bandwidth of the low pass filter is equal to $\Delta f = 2 f_H / f_S$

Relationship between the filter bandwidth and the weighting coefficient

The first generation

$$r \approx \frac{1}{\pi \Delta f} \frac{\sqrt{\sqrt{2}-1}}{\sqrt{1-\left(\cos\left(\omega_C T_S\right)\right)^2}}$$

T_S – the sampling interval,
ω_C – the filter central frequency,
Δf – the relative bandwidth

$$\Delta f = 2\left(f_H - f_S\right)/ f_S$$

The second generation

Number of poles	Weighting coefficient r as the exact function of the filter relative bandwidth Δf	Approximation
1	$r = \sqrt{\dfrac{\sqrt{2}-1}{2\left(1-\cos\left(\pi\Delta f\right)\right)}}$	$r \approx \dfrac{0.2048624}{\Delta f}$
2	$r = \sqrt{\dfrac{\sqrt{2}-1}{6-8\cos\left(\pi\Delta f\right)+2\cos\left(2\pi\Delta f\right)}}$	$r \approx \dfrac{0.0652097315}{\Delta f^{2}}$
3	$r = \sqrt{\dfrac{\sqrt{2}-1}{20-30\cos\left(\pi\Delta f\right)+12\cos\left(2\pi\Delta f\right)-2\cos\left(3\pi\Delta f\right)}}$	$r \approx \dfrac{0.020756902}{\Delta f^{3}}$

$$\cos\left(\pi\Delta f\right) \approx \left(1-\left(\pi\Delta f\right)^2/2\right)$$

The relative bandwidth is defined as $\Delta f = 2 f_H / f_S$

See [Tůma2005]

MATLAB M-files

First generation

```
function x = MyVoldKalman1(y,dt,f,r)

c   = 2*cos(2*pi*f*dt);
N   = max(size(y)); N2 = N-2;
e   = ones(N2,1);
A   = spdiags([e -2*c(1:N2) e],0:2,N2,N);
AA   = r*r*A'*A +speye(N,N);
x    = AA\y;
```

```
Sparse matrix functions
speye – identity matrix
spdiags – diagonal matrix
\ - left matrix divide
```

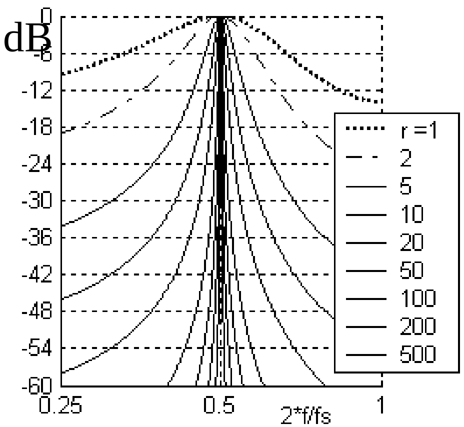
Second generation

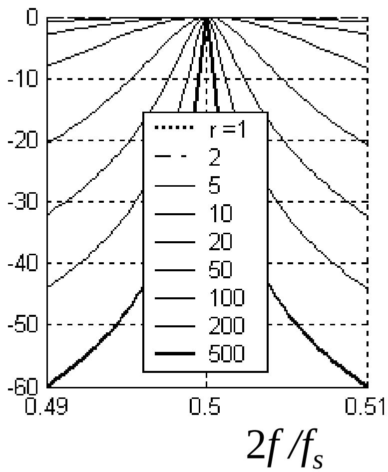
```
function x = MyVoldKalman2(y,dt,f,r,filtord)

N   = max(size(y));
if filtord==1, NR = N-2; else NR = N-3; end;
e   = ones(NR,1);
if filtord==1,
   A = spdiags([e –2*e e],0:2,NR,N);
else
   A = spdiags([e –3*e 3*e -e],0:3,NR,N);
end;
AA = r*r*A'*A +speye(N); yy = exp(-j*2*pi*cumsum(f)*dt).*y;
x   = 2*AA\yy;
```

Effect of the weighting coefficient value on filter selectivity

The frequence response $H(jf)$ of the V-K filter

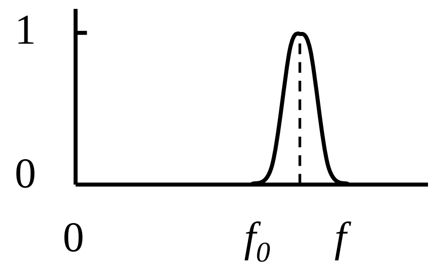
The first generation filter



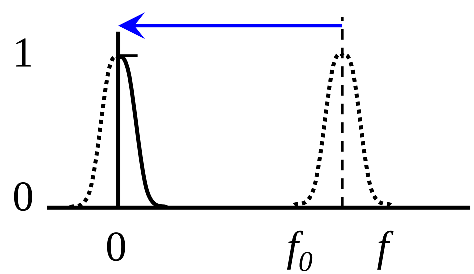


The second generation filter

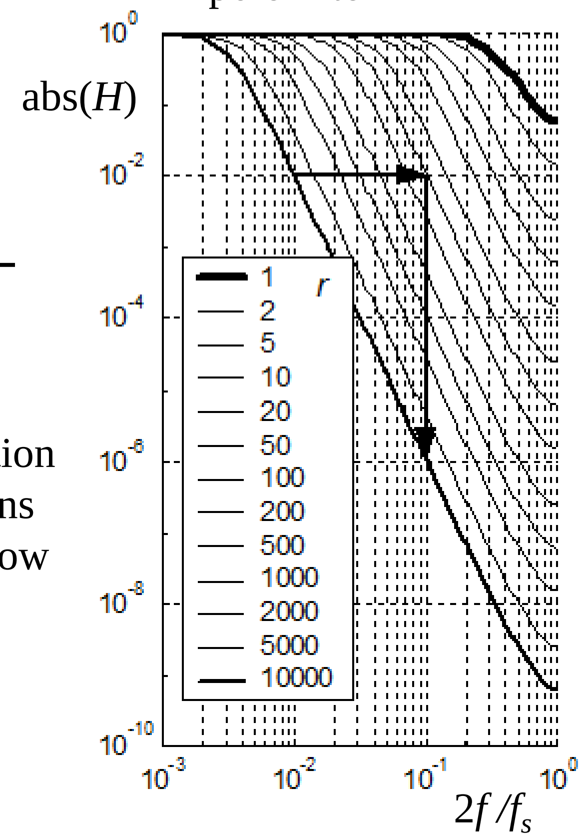
Band-pass Filter



Multiplication of the equation right side by exp($-j\omega_0 t$) turns the band pass filter to the low pass filter

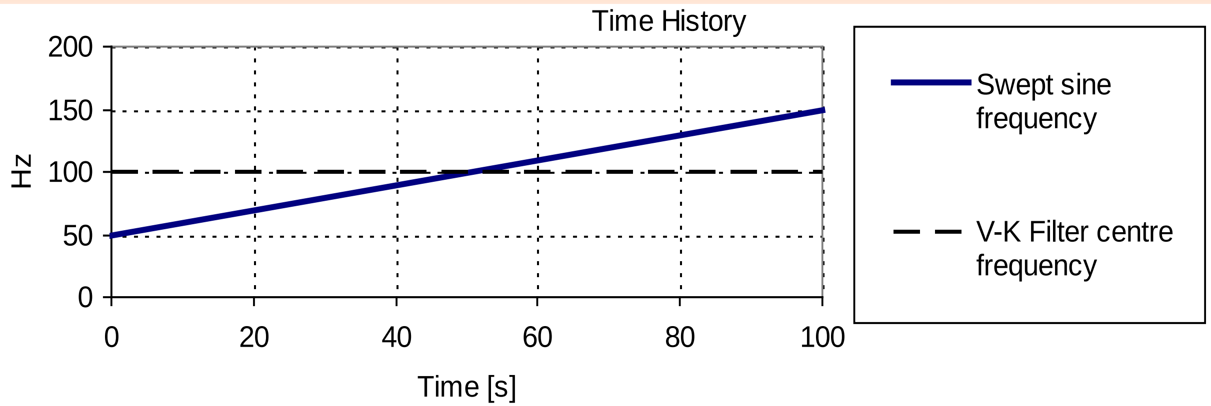


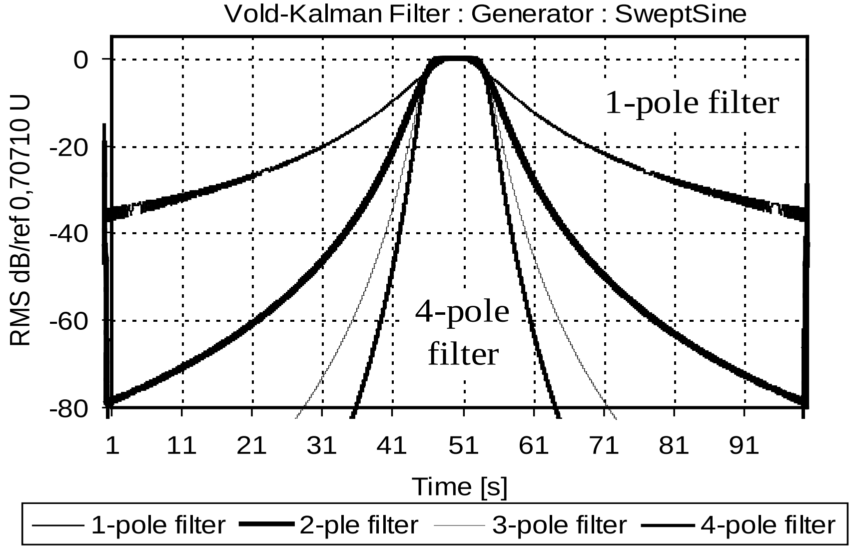
2-pole filter



The low-pass filter roll-off = -40 dB * pole number

Example no.1: VK-Filter frequency response



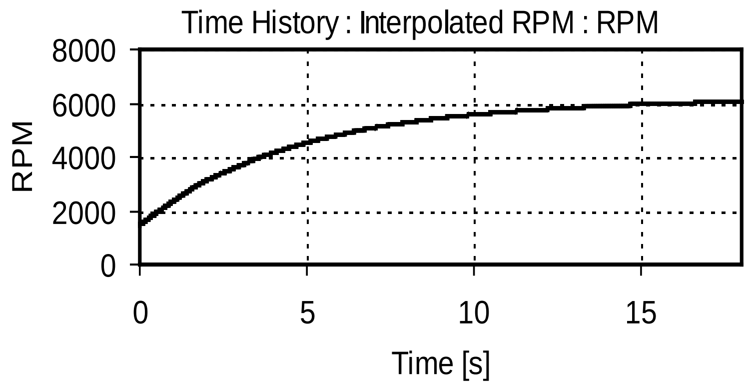


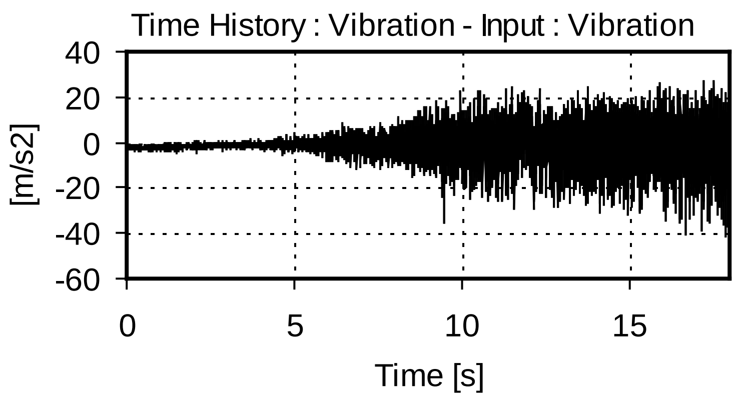
The second generation of the V-K filter

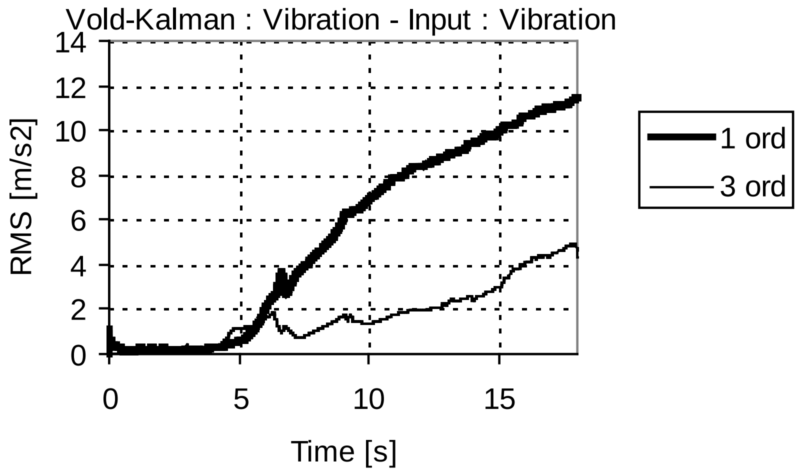
Signal Analyser, MATLAB

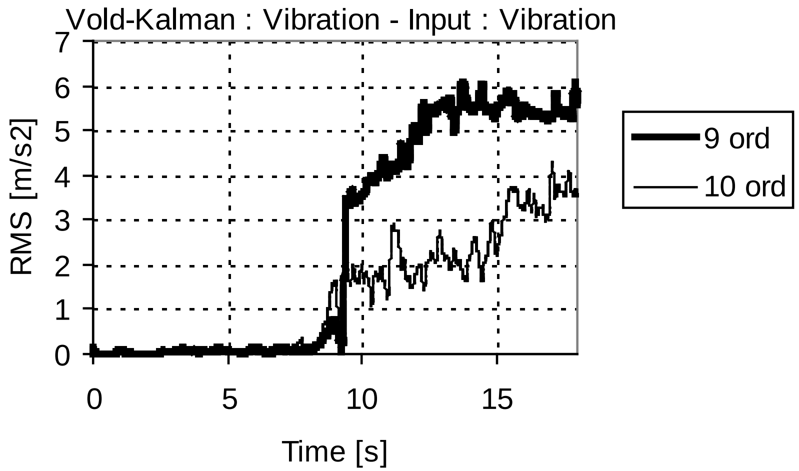
Example no.2: Run-up of a motor

73728 samples = 73728 equations









Multicomponent filtration

For extraction of P components from the measured signal, the objective function is as follows

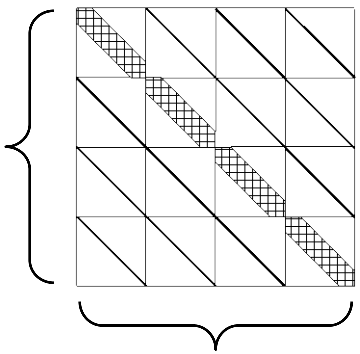
$$J = r^2\left(\boldsymbol{\varepsilon}_1^T\boldsymbol{\varepsilon}_1 + \ldots + \boldsymbol{\varepsilon}_P^T\boldsymbol{\varepsilon}_P\right) + \boldsymbol{\eta}^T\boldsymbol{\eta} \rightarrow \min$$

The global minimum of the objective function is resulting from solving the linear system of equations

$$\frac{\partial J}{\partial \mathbf{x}_i^H} = \left(\mathbf{B}_i + \mathbf{E}\right)\mathbf{x}_i + \mathbf{C}_i^H \sum_{\substack{k=1\\k\neq i}}^{P} \mathbf{C}_k\mathbf{x}_k - \mathbf{C}_i^H\mathbf{y} = \mathbf{0}, \quad i = 1,\ldots,P \qquad \left(\mathbf{C}_i^H\mathbf{C}_j = \mathbf{E},\ i \neq j\right)$$

where **E** is the unity matrix, $\mathbf{B}_i = r^2\mathbf{A}_i^T\mathbf{A}_i$, for the 1st generation is $\mathbf{C}_i = \mathbf{E}$ and for the 2nd generation $\mathbf{A}_i = \mathbf{A}$

P matrix blocks



P matrix blocks

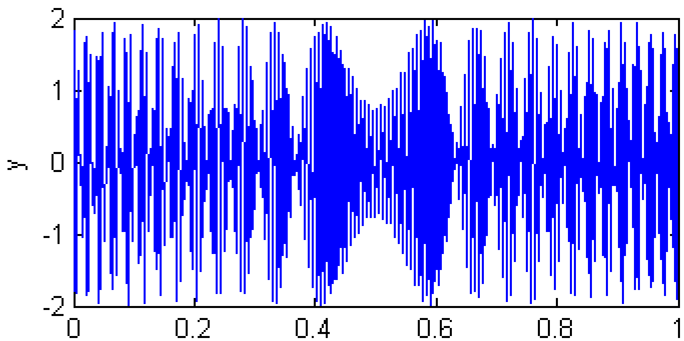
$$\underbrace{\begin{pmatrix} \mathbf{B}_1+\mathbf{E} & \mathbf{C}_1^H\mathbf{C}_2 & \ldots & \mathbf{C}_1^H\mathbf{C}_P \\ \mathbf{C}_2^H\mathbf{C}_1 & \mathbf{B}_2+\mathbf{E} & \ldots & \mathbf{C}_2^H\mathbf{C}_P \\ \ldots & \ldots & \ldots & \ldots \\ \mathbf{C}_P^H\mathbf{C}_1 & \mathbf{C}_P^H\mathbf{C}_2 & \ldots & \mathbf{B}_P+\mathbf{E} \end{pmatrix}}_{P\text{x}P\text{–block matrix}} * \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \ldots \\ \mathbf{x}_P \end{pmatrix} = \begin{pmatrix} \mathbf{C}_1^H\mathbf{y} \\ \mathbf{C}_2^H\mathbf{y} \\ \ldots \\ \mathbf{C}_P^H\mathbf{y} \end{pmatrix} \qquad \mathbf{B}\,\mathbf{x}_\Sigma = \mathbf{b}$$

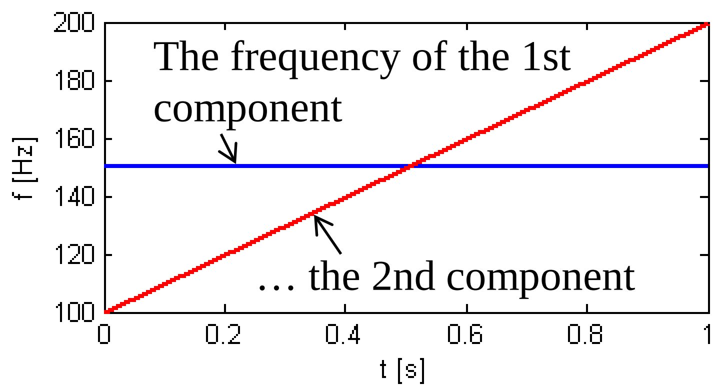
The large-scale system of linear equations $\mathbf{B}\mathbf{x}_\Sigma = \mathbf{b}$ is solved by using the Preconditioned Conjugate Gradients (PCG) method. This method combines the iterative solution of $\mathbf{B}\,\mathbf{M}^{-1}\,\mathbf{u} = \mathbf{b}$ and $\mathbf{x}_\Sigma = \mathbf{M}^{-1}\mathbf{u}$ where **M** is a preconditioner matrix, which is easily inverted. The iterative part requires the initial guess of $\mathbf{x}_\Sigma$

See [Feldbauer& Holdrich]

Example no.3: PCG algorithm in decomposition of two combined signals

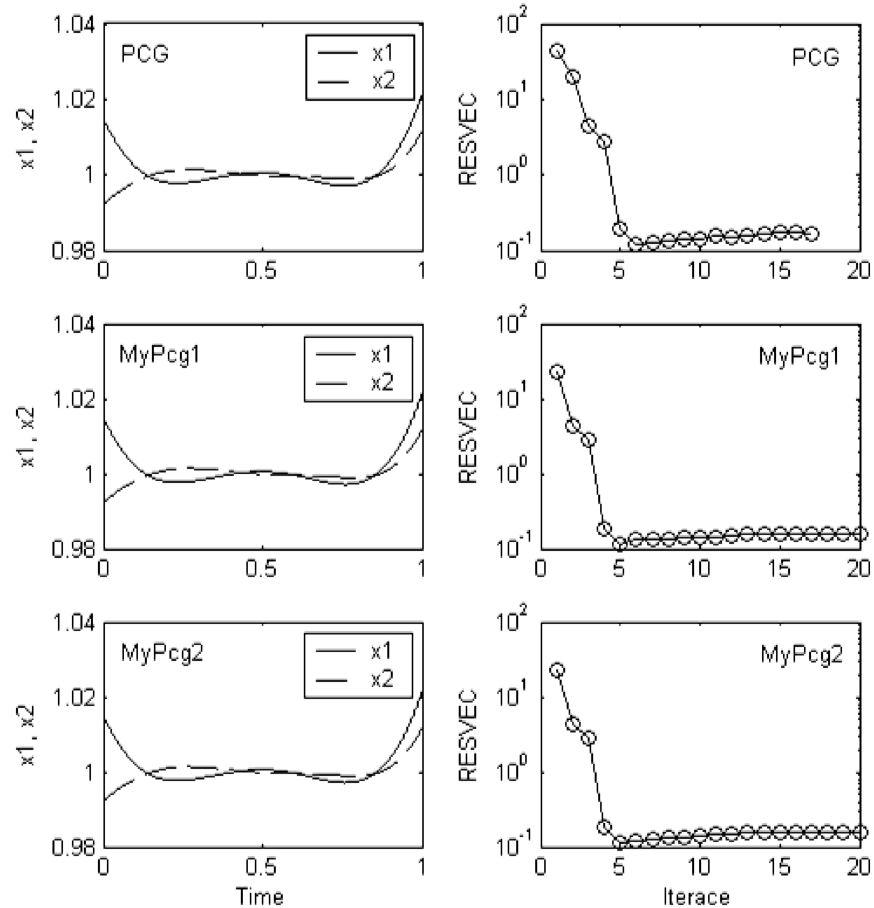
The sum of two harmonic signals with the same unity amplitude





1000 samples * components = 2000 equations

Envelope | Residuum



Time | Iterations

David Hilbert (January 23, 1862 – February 14, 1943) was a German mathematician, recognized as one of the most influential and universal mathematicians of the 19th and early 20th centuries. He discovered and developed a broad range of fundamental ideas in many areas, including invariant theory and the axiomatization of geometry. He also formulated the theory of Hilbert spaces, one of the foundations of functional analysis.

ANALYTIC SIGNALS AND HILBERT TRANSFORM

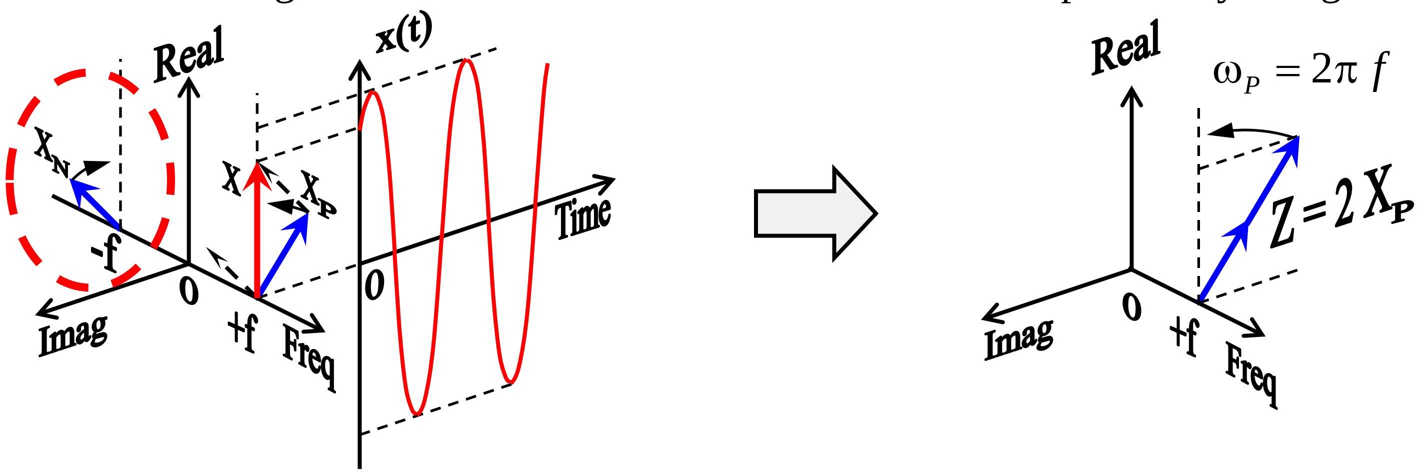
Analytic signals

The analytic signal is a complex signal with an imaginary part, which is the Hilbert transform of the signal real part. The decomposition of a real signal into harmonic components results in the sum of harmonic functions. Each of this function can be decomposed into the pair of the phasors, which are rotating in the opposite direction. The analytic signal creates the phasors rotating in the positive direction.

The analytic signal is a tool for amplitude and phase demodulation of the modulated harmonic signals.

Real harmonic signal

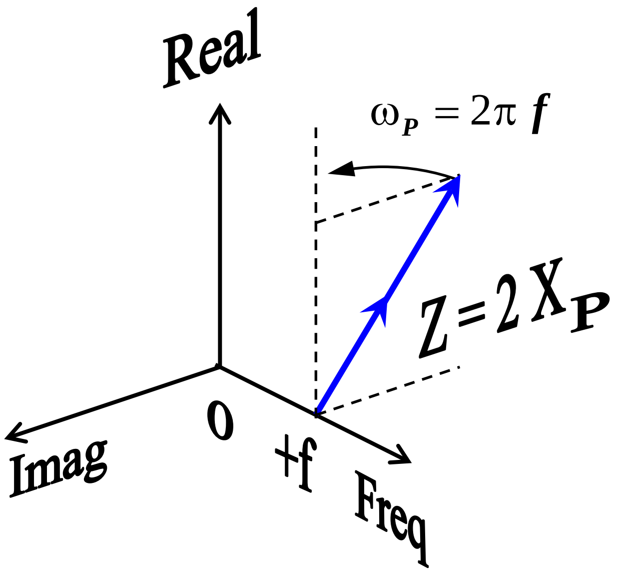
Complex analytic signal

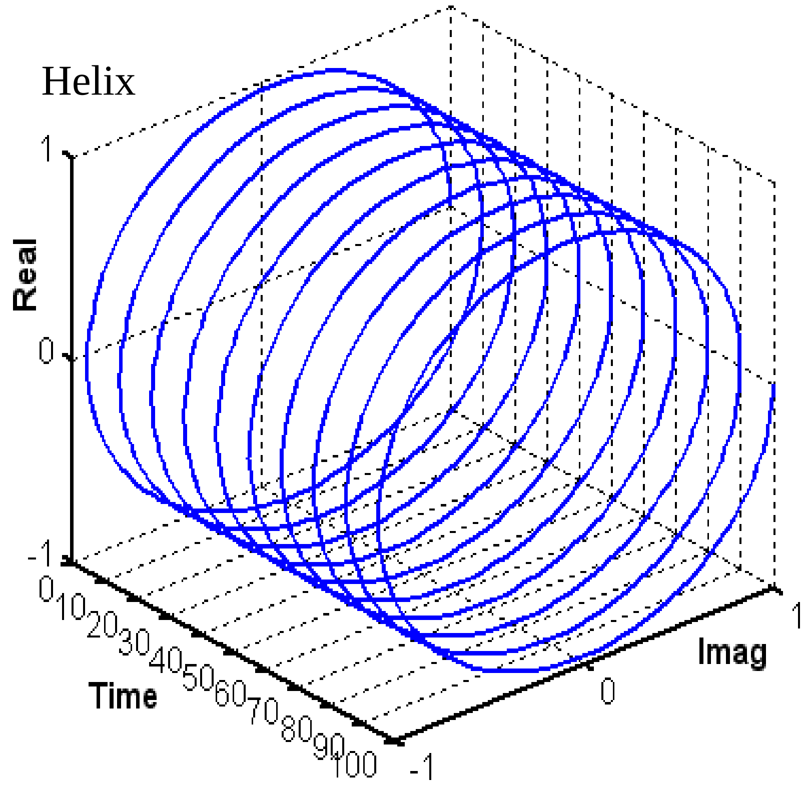


To obtain the analytical signal the phasor $\mathbf{X_N}$ has to be removed and the phasor $\mathbf{X_P}$ has to be multiplied by 2.

Analytic signals in the 3D-space

The position vector is rotating in the complex plane. If the 2D space is extended to 3D space with the third axis as a time axis then the vector end point moves on the helix trajectory.

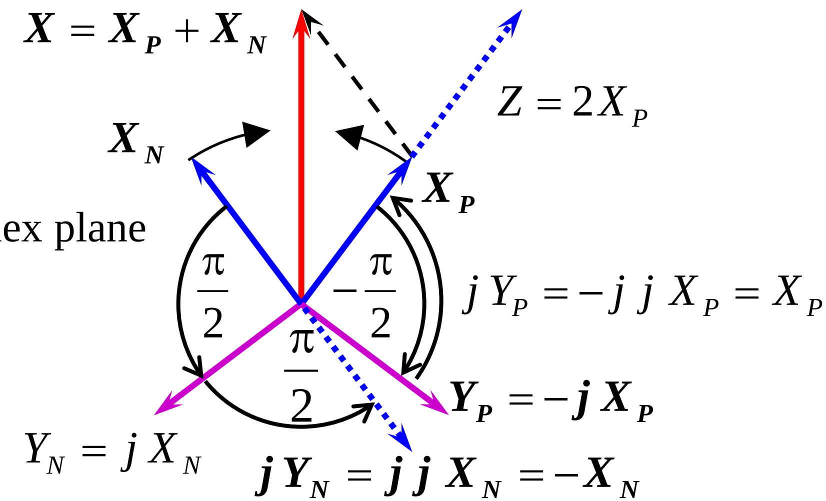




Analytic signals and the Hilbert transform

The complex position vectors as phasors, which are corresponding to a harmonic signal.

The complex plane



Evaluation of the Hilbert transform using …

- Fast Fourier Transform (FFT)
- Digital filters

Time signal **+ *j*** Hilbert transform **=** Analytic signal

$$\boldsymbol{X} = \boldsymbol{X}_P + \boldsymbol{X}_N$$

$$\begin{aligned} Y &= Y_P + Y_N = \\ &= -j\,X_P + j\,X_N = \\ &= -j\left(X_P - X_N\right) \end{aligned}$$

$$\begin{aligned} Z &= X_P + X_N + j\left(Y_P + Y_N\right) = \\ &= X_P + X_N + \left(X_P - X_N\right) = \\ &= 2X_P \end{aligned}$$

The phasors ***Y*** associated to the Hilbert transform of a pair of phasors ***X*** are obtained by rotation these phasors by the angle of +/- π/2 radians.

Definition of the Hilbert transform as the Cauchy principal value integral

The Cauchy principal value (*P.V.*) expands the class of certain improper integrals for which the finite integral exists as for example the integral

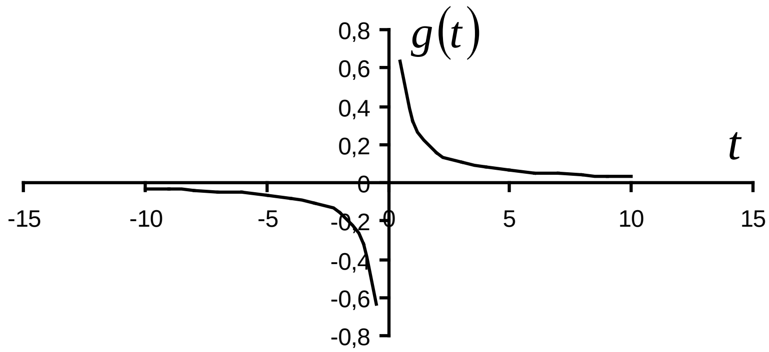
$$\lim_{\varepsilon\to 0+}\left[\int_a^{\xi-\varepsilon} f(x)\,\mathrm{d}\,x + \int_{\xi+\varepsilon}^{b} f(x)\,\mathrm{d}\,x\right] \quad \text{where} \quad \xi\in(a,b),\ \int_a^{\xi} f(x)\,\mathrm{d}\,x = \pm\infty,\ \int_{\xi}^{b} f(x)\,\mathrm{d}\,x = \mp\infty$$

The Hilbert Transform can be defined as the principal value integral

$$y(t) = \frac{1}{\pi} P.V. \int_{-\infty}^{+\infty} \frac{x(\tau)}{t-\tau}\,\mathrm{d}\,\tau$$

Let $x(t)$ be an impulse Dirac function $\delta(t)$, then the Hilbert transformer impulse response is as follows

$$g(t) = \frac{1}{\pi} P.V. \int_{-\infty}^{+\infty} \frac{\delta(\tau)}{t-\tau}\,\mathrm{d}\,\tau = \frac{1}{\pi t}$$



The Hilbert transform as a transfer function

The Hilbert transform of a time function to another time function can be described by the standard transfer function in the frequency domain. Let $X(j\omega)$ and $Y(j\omega)$ be the Fourier transform of the original continuous time signal $x(t)$ into the same signal $y(t)$, respectively.

$$H_{HT}(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \begin{cases} -j, & \omega > 0 \\ j, & \omega < 0 \end{cases}$$

To turn the non-decaying function to the decaying function, let the frequency transfer function be extended

$$H(j\omega) = \begin{cases} -je^{-\sigma\omega}, & \omega > 0 \\ je^{\sigma\omega}, & \omega < 0 \end{cases} \qquad \lim_{\sigma\to 0} H(j\omega) = H_{HT}(j\omega)$$

The impulse response is an inverse Fourier transform of the frequency transfer function

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} H(j\omega)e^{j\omega t}\,\mathrm{d}\omega = \frac{1}{2\pi}\left[j\int_{-\infty}^{0} e^{\sigma\omega + j\omega t}\,\mathrm{d}\omega - j\int_{0}^{+\infty} e^{-\sigma\omega + j\omega t}\,\mathrm{d}\omega \right] =$$

$$= \frac{j}{2\pi}\int_{0}^{+\infty}\left(e^{-\sigma\omega - j\omega t} - e^{-\sigma\omega + j\omega t}\right)\mathrm{d}\omega = \frac{j}{2\pi}\left[-\frac{1}{\sigma + jt}e^{-(\sigma + jt)\omega} + \frac{1}{\sigma - jt}e^{-(\sigma - jt)\omega} \right]_0^{\infty} = \frac{t}{\pi\left(\sigma^2 + t^2\right)}$$

$$g_{HT}(t) = \lim_{\sigma\to 0} g(t) = \lim_{\sigma\to 0}\frac{t}{\pi\left(\sigma^2 + t^2\right)} = \frac{1}{\pi t}$$

See [http://w3.msi.vxu.se/exarb/mj_ex.pdf]

Analytic signals and the Hilbert transform of some signals

The envelope and phase of the harmonic signals

Real part	Imaginary part	Envelope	Phase
$x(t)$	$y(t)$	$E(t)$	$\beta(t)$
$A\sin(\omega t)$	$-A\cos(\omega t)$	A	$\omega t-\pi/2$
$A\cos(\omega t)$	$A\sin(\omega t)$	A	ωt

Hilbert transform

Signal	Hilbert Transform
$x(t)$	$y(t)$
$\sin(\omega t)$	$-\cos(\omega t)$
$\cos(\omega t)$	$\sin(\omega t)$
$1/(t^2+1)$	$t/(t^2+1)$
$\sin(t)/t$	$(1-\cos(t))/t$
$\delta(t)$	$1/\pi t$

The use of FFT for computing the Hilbert transform

The algorithm for computing the Hilbert transform is broken down into three steps

i. The Fast Fourier Transform (FFT) of the real input time record to obtain phasors rotating in positive and negative directions

ii. Rotation the phasor $\boldsymbol{X_N}$ in the positive direction by the angle of + π/2 radians and the phasor $\boldsymbol{X_P}$ in the negative direction by the angle of - π/2 radians (exchanging the real and imaginary parts)

iii. The Inverse Fast Fourier Transform (FFT) of the rotated phasors $\boldsymbol{Y_P}$ and $\boldsymbol{Y_N}$ to obtain the Hilbert transform of the input record.

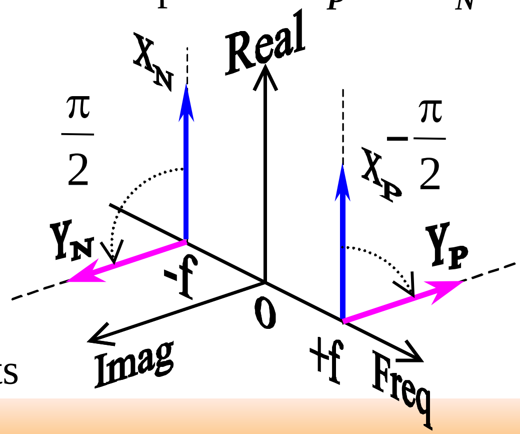
i) $X(j\omega) = FFT\{x(k)\} \quad \Rightarrow$	ii) $X(j\omega) \rightarrow Y(j\omega)$	iii) $\Rightarrow \quad y(k) = IFFT\{Y(j\omega)\}$

Diagram showing how to transform the phasors $\boldsymbol{X_P}$ and $\boldsymbol{X_N}$ to the phasors $\boldsymbol{Y_P}$ and $\boldsymbol{Y_N}$

$$\boxed{Y_N = j\,X_N}$$

$$Y_N = j\left(\mathrm{Re}(X_N) + j\,\mathrm{Im}(X_N)\right) = \\ = -\mathrm{Im}(X_N) + j\,\mathrm{Re}(X_N)$$

Exchanging the real and imag parts



$$\boxed{Y_P = -j\,X_P}$$

$$Y_P = -j\left(\mathrm{Re}(X_P) + j\,\mathrm{Im}(X_P)\right) = \\ = \mathrm{Im}(X_P) - j\,\mathrm{Re}(X_P)$$

Exchanging the real and imag parts

The use of digital filters for computing the Hilbert transform

Let $X(e^{j\omega TS})$ and $Y(e^{j\omega TS})$ be the Fourier transform of the original sample sequence x_t into the sample sequence y_t, respectively.

Frequency response function

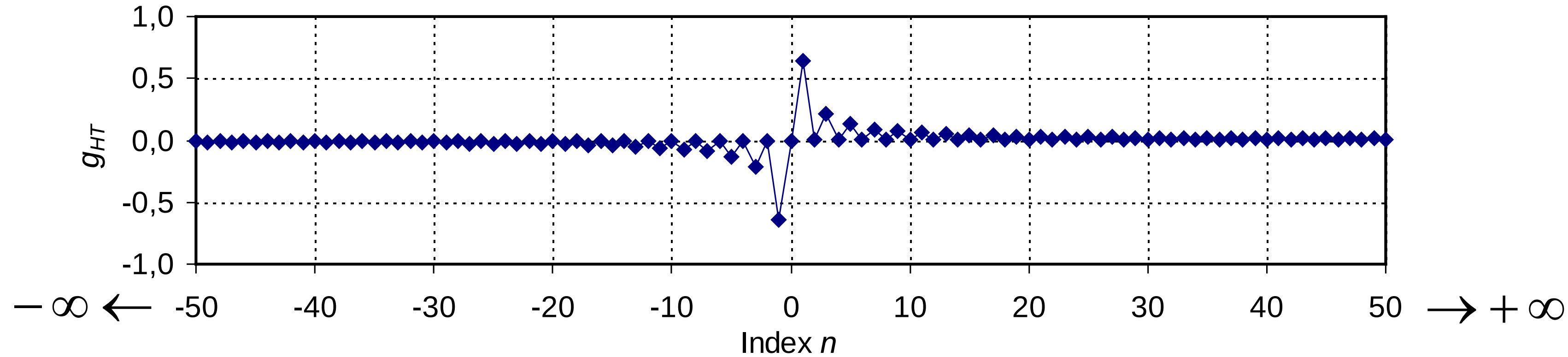
$$G_{HT}\left(e^{j\omega T_S}\right)=\frac{Y\left(e^{j\omega T_S}\right)}{X\left(e^{j\omega T_S}\right)}=\begin{cases}-j, & +\pi>\omega T_S>0\\ j, & -\pi<\omega T_S<0\end{cases}$$

Impulse response

$$g_{HT}(n)=\frac{1}{2\pi}\int_{-\pi}^{+\pi}G_{HT}\left(e^{j\omega T_S}\right)e^{j\omega T_S n}\,\mathrm{d}(\omega T_S)=$$

$$=\begin{cases}0, & n=2k\\ 2/\pi n, & n=2k+1\end{cases}$$

Impulse Response of the Ideal Hilbert transformer



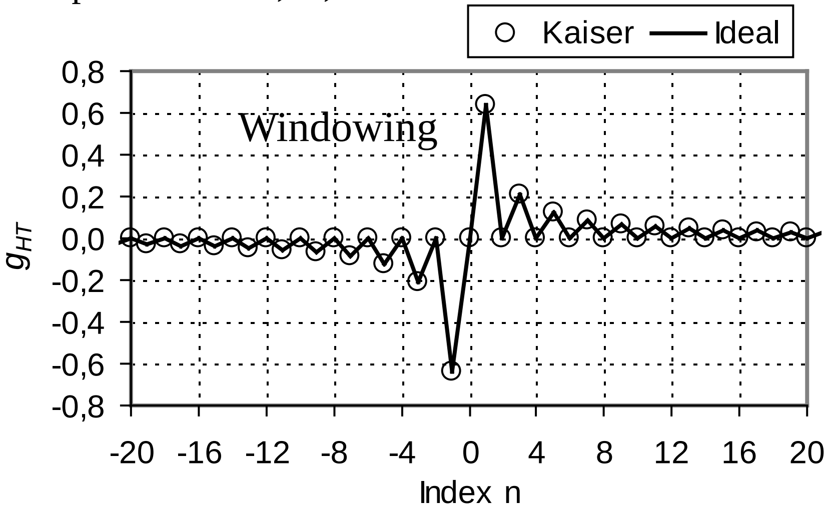
The nonzero response for the negative index n means that the impulse response corresponds to a non-causal system. Response precedes the change at the system input.

Hilbert transformer filters

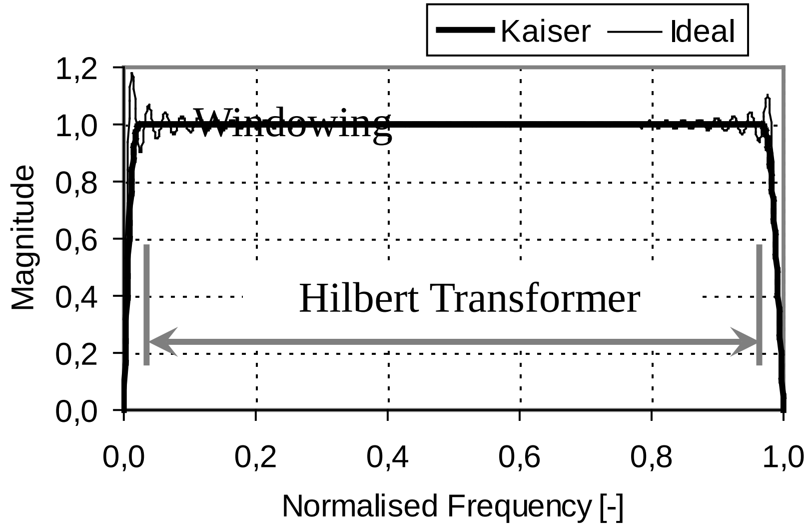
The impulse response of FIR filters is the same as these filters non-zero coefficients. If the infinite impulse response is shorten to a finite number of non-zero samples then the this response will corresponds to a FIR filter. Due to the linearity of the filter phase the symmetric or anti-symmetric coefficients are preferred. As in the case of FIR filter the impulse response has to be delayed in such a way that the impulse response of the non-causal system is changed to the response of the causal system. The filter is called as a Hilbert transformer or a 90-degree phase shifter.

The digital filter acts as a Hilbert transformer only for a frequency band in which the magnitude of the frequency response function is equal to unit. The impulse response which is corrected with the use of the Kaiser window smooths the frequency response function.

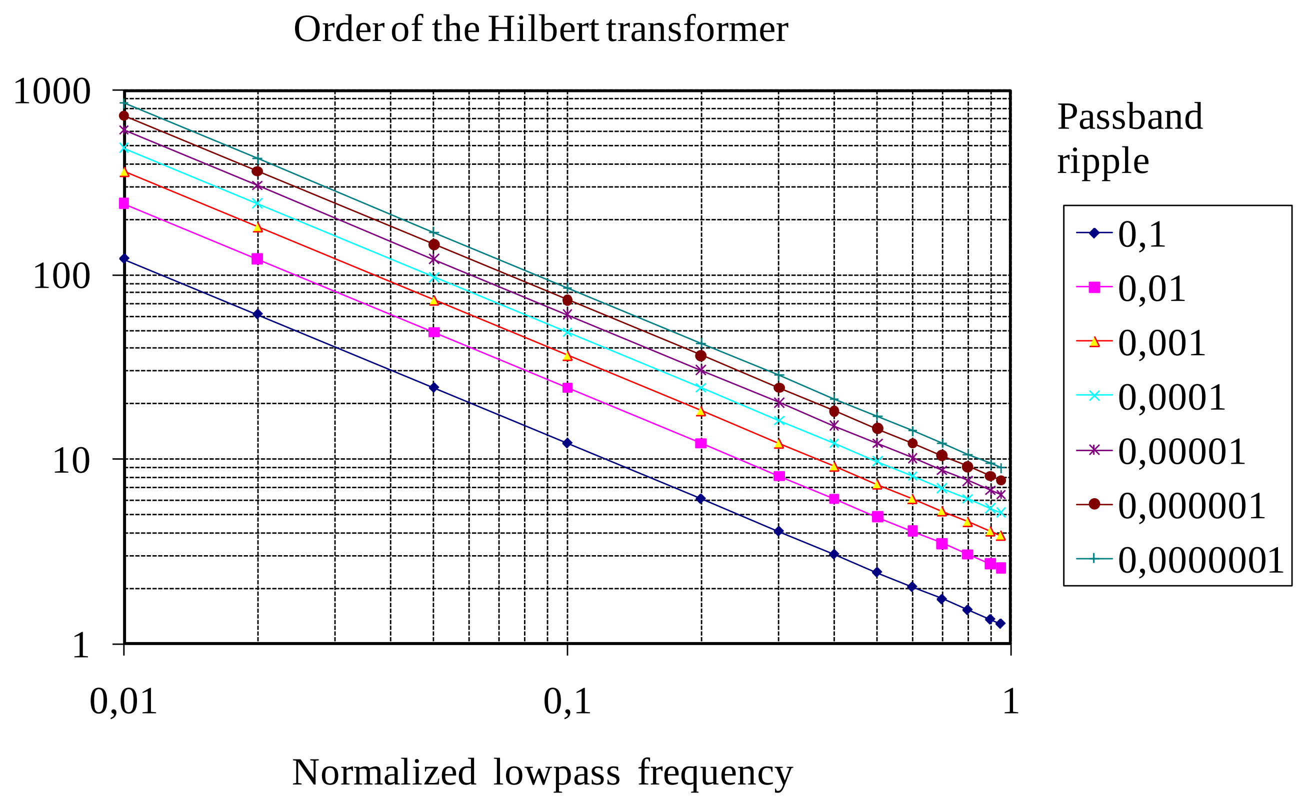
The 160-order FIR filter with the finite impulse response n = -80,…,+80



Frequency response function of the Hilbert transformer



Minimum order of the Hilbert Transformer as a FIR filter



Creation analytical signals with the use of the digital filter

Consider the complex analytic signal z_t composed of the real part x_t and its Hilbert transform as the imaginary part y_t $\quad z_t = x_t + jy_t$

The discrete Fourier transform of the sample sequences is as follows

$$Z\left(e^{j\omega T_S}\right) = X\left(e^{j\omega T_S}\right) + jY\left(e^{j\omega T_S}\right)$$

and its conjugate symmetric formula

$$Z^*\left(e^{-j\omega T_S}\right) = X^*\left(e^{-j\omega T_S}\right) - jY^*\left(e^{-j\omega T_S}\right)$$

Two previous formulas may be added together or subtracted each other. It results in

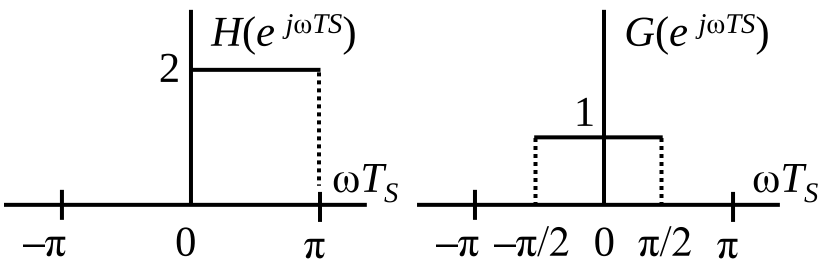
$$X\left(e^{j\omega T_S}\right) = \left(Z\left(e^{j\omega T_S}\right) + Z^*\left(e^{-j\omega T_S}\right)\right)/2$$

$$jY\left(e^{j\omega T_S}\right) = \left(Z\left(e^{j\omega T_S}\right) - Z^*\left(e^{-j\omega T_S}\right)\right)/2$$

If these formulas are added together and since by assumption, $Z(e^{j\omega TS}) = 0$ for $-\pi < \omega T_S < 0$, then the transfer function, relating $Z(e^{j\omega TS})$ to $X(e^{j\omega TS})$, is obtained

$$H\left(e^{j\omega T_S}\right) = \frac{Z\left(e^{j\omega T_S}\right)}{X\left(e^{j\omega T_S}\right)} = \begin{cases} 2, & +\pi > \omega T_S > 0 \\ 0, & -\pi < \omega T_S < 0 \end{cases}$$

This formula confirms the previous result obtained with the use of phasors.
The frequency response $H(e^{j\omega TS})$ of the discrete-time filter is as follows



Consider the half-band lowpass filter with the frequency response $G(e^{j\omega TS})$, which is obtained by shifting the frequency response $H(e^{j\omega TS})$ by $\pi/2$ radians and scaling by a factor 1/2.

$$G\left(e^{j\omega T_S}\right) = \frac{1}{2} H\left(e^{j\omega T_S + \pi/2}\right) = \begin{cases} 1, & 0 < \left|\omega T_S\right| < \pi/2 \\ 0, & \pi/2 < \left|\omega T_S\right| < \pi \end{cases}$$

The filter $H(e^{j\omega TS})$ is referred to as a complex half-band filter while the filter $G(e^{j\omega TS})$ is referred to as a real half-band filter .

See [Mitra]

FIR complex half-band filters

The relationship between the transfer function of a complex half-band filter $H(z)$ and a real half-band filter $G(z)$ is as follows

$$H(z) = j2G(-jz)$$

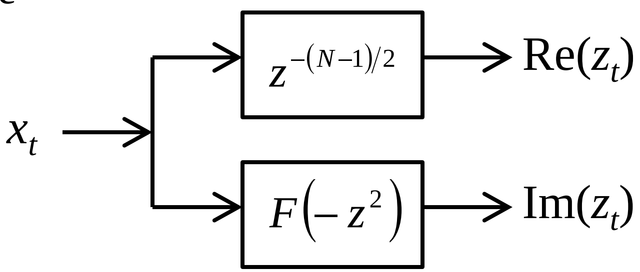
FIR complex half-band filter

Consider a wideband linear-phase filter $F(z)$ of degree $(N-1)/2$ with a passband from 0 to $2\omega_P$, a transition band from $2\omega_P$ to π, and a passband ripple 2δ. Since $(N-1)/2$ is odd, $F(z)$ has a zero at $z = -1$. Define

$$G(z) = \frac{1}{2}\left[z^{-(N-1)/2} + F\left(z^2\right)\right]$$

$G(z)$ is the desired half-band lowpass filter and has an impulse response

$$g_{HB}(n) = \begin{cases} f(n/2), & n \text{ even} \\ 0, & n \text{ odd}, n \neq (N-1)/2 \\ 1/2, & n = (N-1)/2 \end{cases}$$



FIR realization of a complex half-band filter

where $f(T_S n)$ is the impulse response of $F(z)$.
After substitution, we obtain the FIR complex half-band filter

$$H(z) = j\left[(-jz)^{-(N-1)/2} + F\left(-z^2\right)\right] = z^{-(N-1)/2} + jF\left(-z^2\right)$$

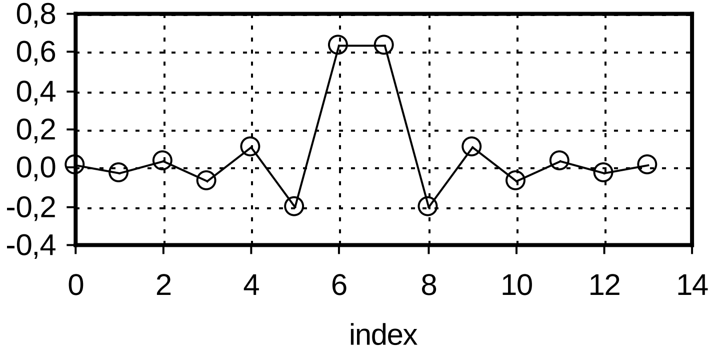
See [Mitra]

The use of Matlab in design of the Hilbert transformer

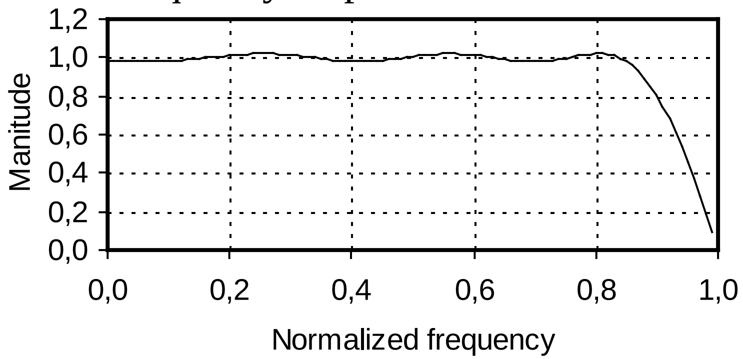
The wideband linear-phase filter $F(z)$ of degree 13

>> b=remez(13,[0 0.85 0.9 1], [1 1 0 0],[2 0.05]) ;

Impulse response of wideband filter



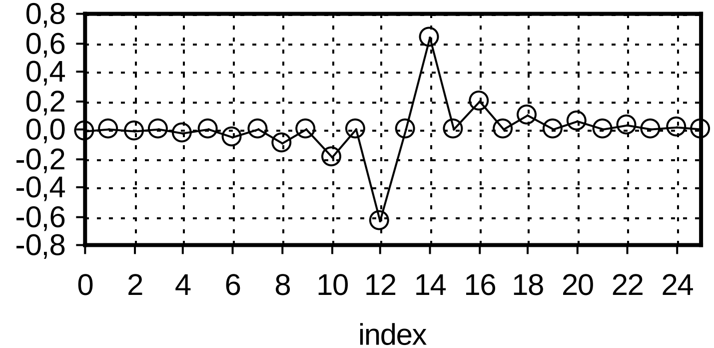
Frequency response of wideband filter



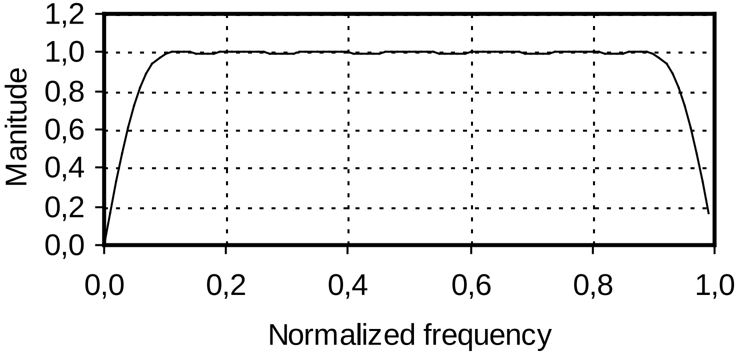
The Hilbert transformer $F(-z^2)$ of degree 26

>> bb=remez(26,[0.1 0.9], [1 1],'Hilbert');

Impulse response of Hilbert transformer



Frequency response of Hilbert transformer



Note that the filter has only 13 non-zero coefficients

IIR complex half-band filters

A large class of stable IIR real coefficient half-band filter of odd order can be expressed as

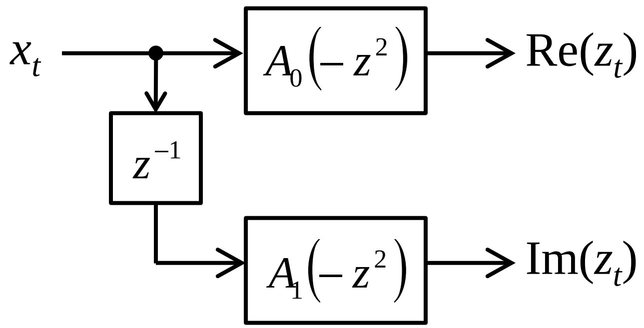
$$G(z) = \frac{1}{2}\left[A_0\left(z^2\right) + z^{-1}A_1\left(z^2\right)\right]$$

where $A_0(z)$ and $A_1(z)$ are stable allpass transfer functions. After substitution,

$$H(z) = j2G(-jz)$$

we obtain the transfer function of a complex half-band filter

$$H(z) = A_0\left(-z^2\right) + jz^{-1}A_1\left(-z^2\right)$$



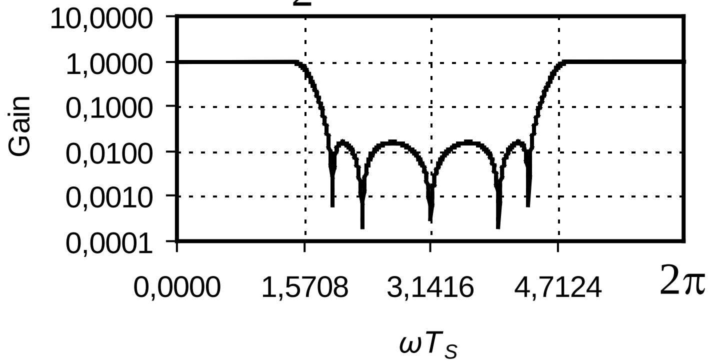
IIR realization of a complex half-band filter

See [Mitra]

The use of Matlab in design of the Hilbert transformer

Frequency response of a real coefficient half-band filter designed for $\omega_P = 0.4\pi$, $\omega_S = 0.6\pi$, $\delta_P = 0.0155$

$$G(z) = \frac{1}{2}\left[A_0\left(z^2\right) + z^{-1}A_1\left(z^2\right)\right]$$



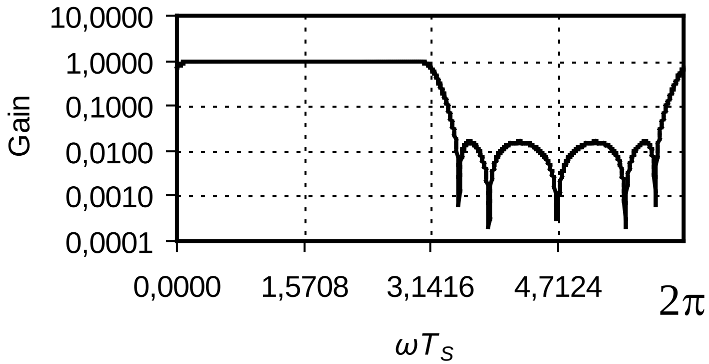
$A_0(z)$ and $A_1(z)$ are allpass filters of order 1

$$A_0(z) = \frac{0.2368041466 + z^{-1}}{1 + 0.2368041466\, z^{-1}}$$

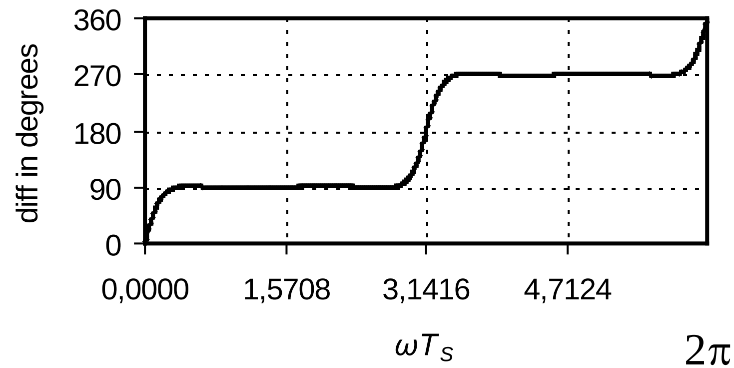
$$A_1(z) = \frac{0.7149039978 + z^{-1}}{1 + 0.7149039978\, z^{-1}}$$

Frequency response of a complex half-band filter

$$H(z) = A_0\left(-z^2\right) + jz^{-1}A_1\left(-z^2\right)$$

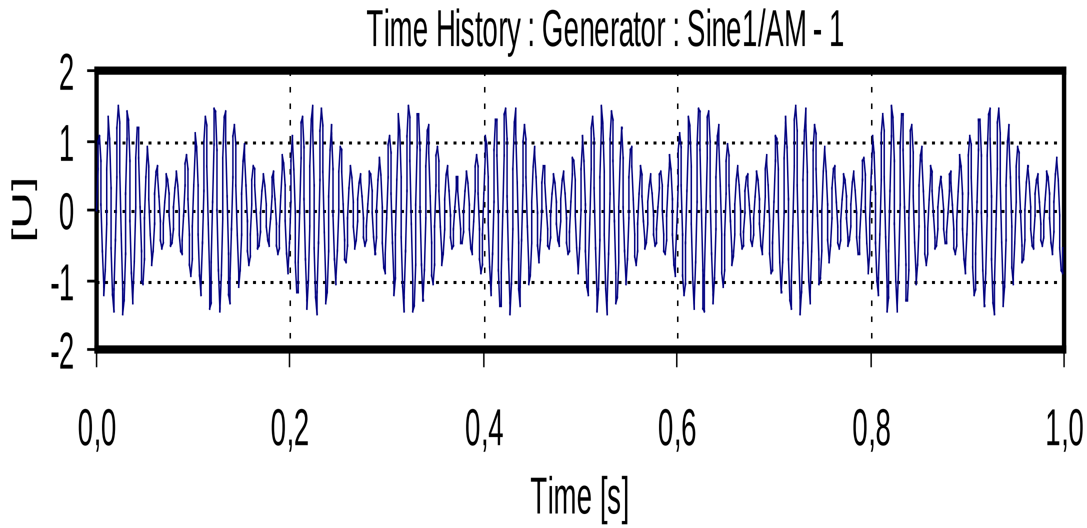


Phase difference between the allpass functions $A_0(-z^2)$ and $A_1(-z^2)$.



Note that the phase difference is 90^0 for the positive frequency band and 270^0 for the negative frequency band .

See [Mitra]



HARMONIC SIGNAL MODULATION

Modulation of harmonic signals

Nomenclature

- Carrying component ………………….. $x_0(t) = A\cos(\omega_0 t + \varphi_0)$
  (harmonic signal without modulation)
  Amplitude A
  Phase $\varphi(t) = \omega_0 t + \varphi_0$
  Initial phase φ_0
- Amplitude modulation signal ………... $x_A(t) = \beta_{AM}\cos(\omega_{AM} t + \varphi_{AM})$
- Phase modulation signal ………………... $x_P(t) = \beta_{PM}\cos(\omega_{PM} t + \varphi_{PM})$
- Mixed modulation (amplitude and phase)

Modulated signal   Modulation signals   Carrying component phase

$$x(t) = A\underbrace{\left(1 + \overbrace{\beta_{AM}\cos(\omega_{AM} t + \varphi_{AM})}\right)}_{\text{Amplitude}}\cos\underbrace{\left(\overbrace{\omega_0 t + \varphi_0} + \overbrace{\beta_{PM}\cos n(\omega_{PM} t + \varphi_{PM})}\right)}_{\text{Phase}}$$

See [Tůma, 1997]

Amplitude modulation of harmonic signals

Modulated signal Modulation signal Carrying component

$$x(t) = A\big(1+\overbrace{\beta_{AM}\cos(\omega_{AM}t+\varphi_{AM})}\big)\cos(\omega_0 t)$$

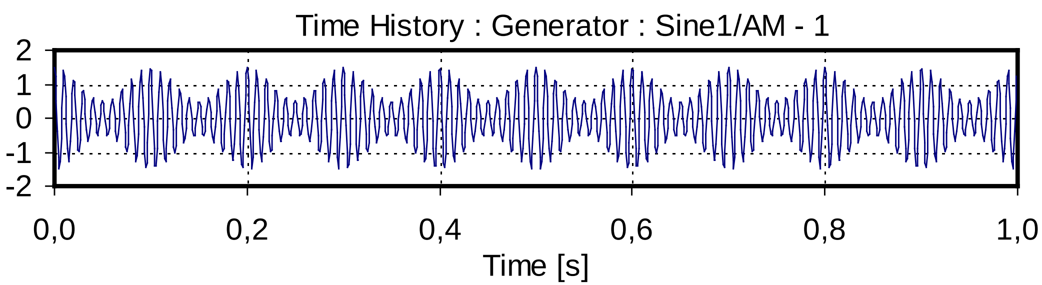
β_{AM} modulation index

modulation frequency

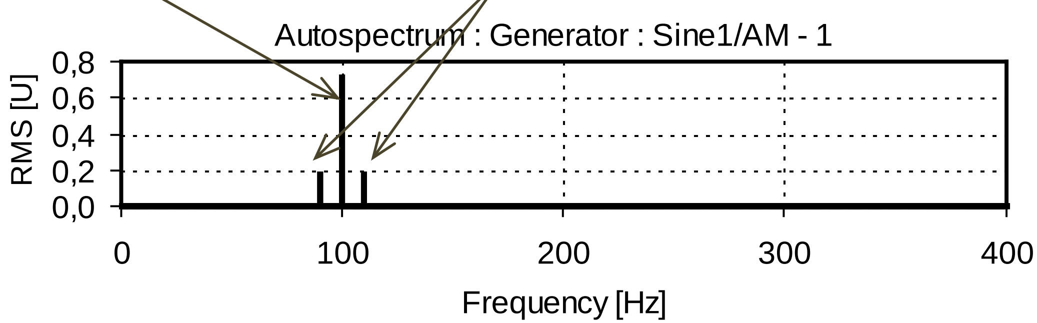
$\omega_{AM} = 2\pi f_{AM}$

carrying frequency

$\omega_0 = 2\pi f_0$



Carrying component Sideband components f_0+f_{AM}, f_0-f_{AM}



Spectrum of the amplitude-modulated signal

Decomposition of the modulated signal on the carrying component and its sidebands

$$
\begin{aligned}
x(t) &= A(1+\beta_{AM}\cos(\omega_{AM}t+\varphi_{AM}))\cos(\omega_0 t) = \\
&= A\cos(\omega_P t) + A\beta_{AM}\cos(\omega_{AM}t+\varphi_{AM})\cos(\omega_0 t) = \\
&= A\cos(\omega_0 t) + A\beta_{AM}/2\left(\cos((\omega_0-\omega_{AM})t-\varphi_{AM}) + \cos((\omega_0+\omega_{AM})t+\varphi_{AM})\right)
\end{aligned}
$$

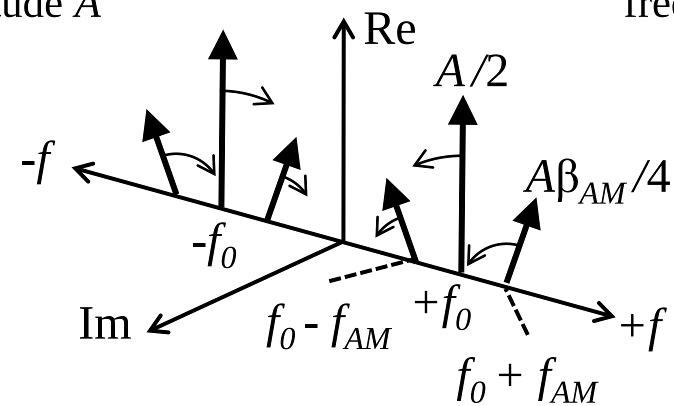
$$
x(t) = \underbrace{A\sin(\omega_0 t)} + \underbrace{A\beta_{AM}/2\ \cos((\omega_0-\omega_{AM})t-\varphi_{AM})} + \underbrace{A\beta_{AM}/2\ \cos((\omega_0+\omega_{AM})t+\varphi_{AM})}
$$

Lower sideband component,
frequency $f_0 - f_{AM}$, amplitude $A\beta_{AM}/2$

The carrying component,
frequency f_0, amplitude A

Upper sideband component,
frequency $f_0 + f_{AM}$, amplitude $A\beta_{AM}/2$

Phasor model



Rotational frequency

Phase modulation of harmonic signals

Modulated signal Carrying component Modulation signal

$$x(t) = A\cos\left(\omega_0 t + \overbrace{\beta_{PM}\cos\left(\omega_{PM} t + \varphi_{PM}\right)}\right)$$

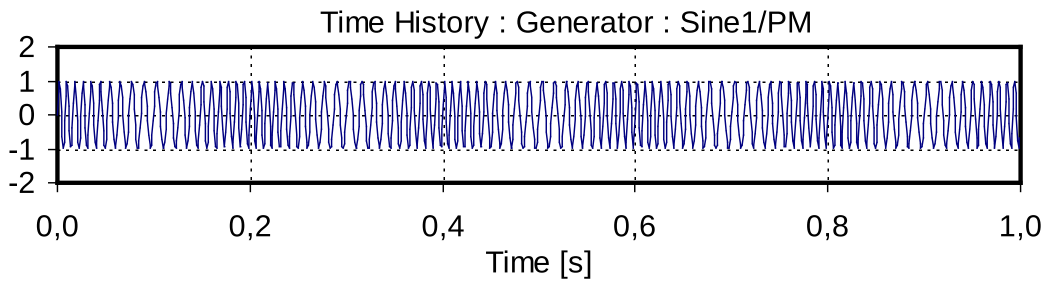
β_{AM} modulation index

modulation frequency

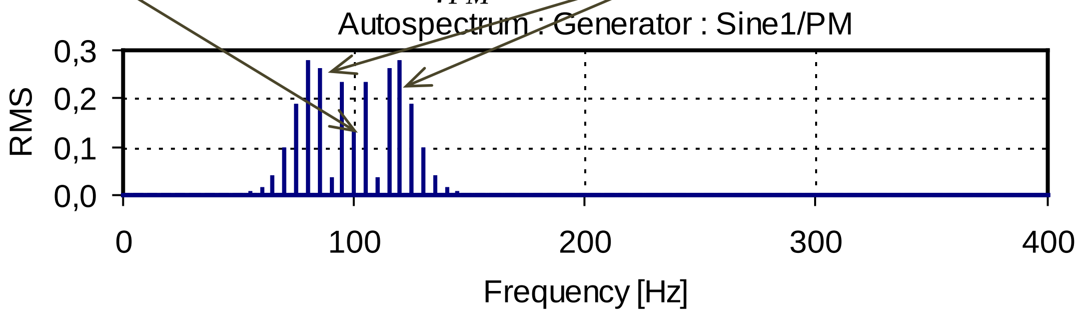
$\omega_{PM} = 2\pi f_{PM}$

carrying frequency

$\omega_0 = 2\pi f_0$



Carrying component The family of the sideband components $f_0 + f_{PM}, \ldots, f_0 - f_{PM}, \ldots$



Spectrum of the phase-modulated signal

Let $x(t) = \cos(\omega_0 t + s_{PM}(t))$ be a phase-modulated signal, where $s_{PM}(t) = \beta_{PM}\cos(\Phi)$, $\Phi = \omega_{PM} t + \varphi_{PM}$

$$x(t) = p_+(t) + p_-(t)$$

$$p_+(t) = \frac{1}{2}\exp(j(\omega_0 t + \beta_{PM}\cos\Phi)) = \frac{1}{2}\exp(j\omega_0 t)\exp(j\beta_{PM}\cos\Phi)$$

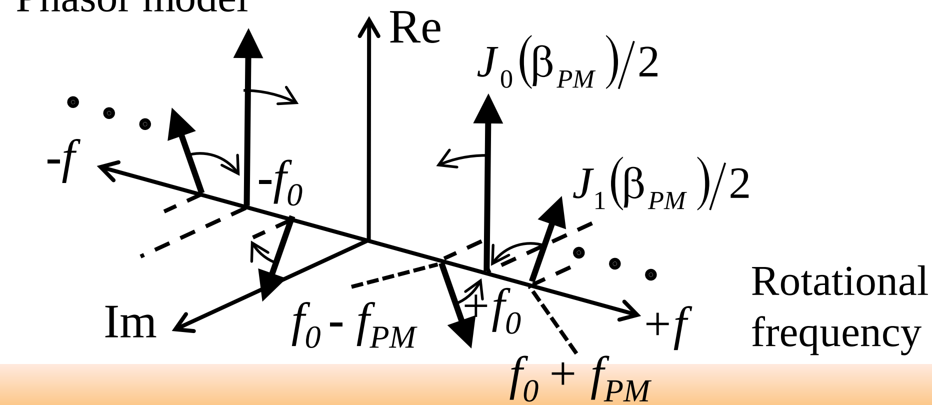
$$p_+(t) = \frac{1}{2}\left(J_0(\beta_{PM})\exp(j\omega_0 t) + \sum_{i=1}^{+\infty} J_i(\beta_{PM}) j^i \left(\exp(j(\omega_0 t + i\Phi)) + \exp(j(\omega_0 t - i\Phi))\right)\right)$$

where $J_i(\beta)$ is the Bessel function of the first kind, for integer orders $i = 0, 1, 2, \ldots$

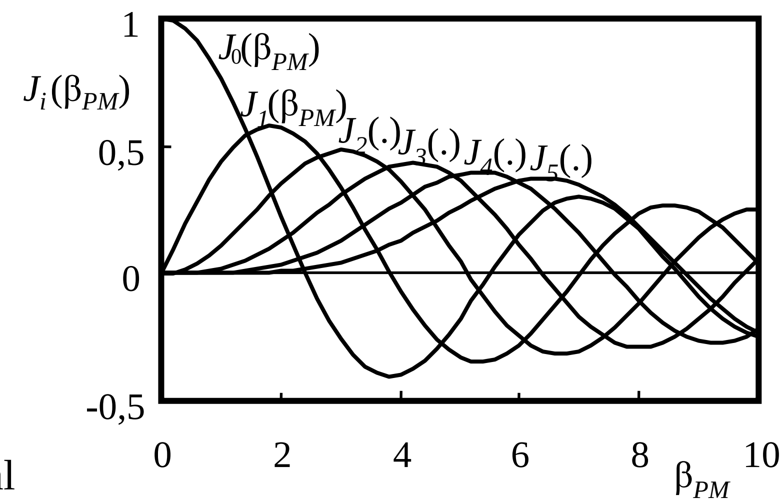
$$J_i(\beta_{PM}) = \sum_{k=0}^{+\infty}\left(\frac{\beta_{PM}}{2}\right)^{2k+i}\frac{(-1)^k}{k!(k+i)!}$$

$$J_{-i}(\beta_{PM}) = (-1)^i J_i(\beta_{PM})$$

Phasor model

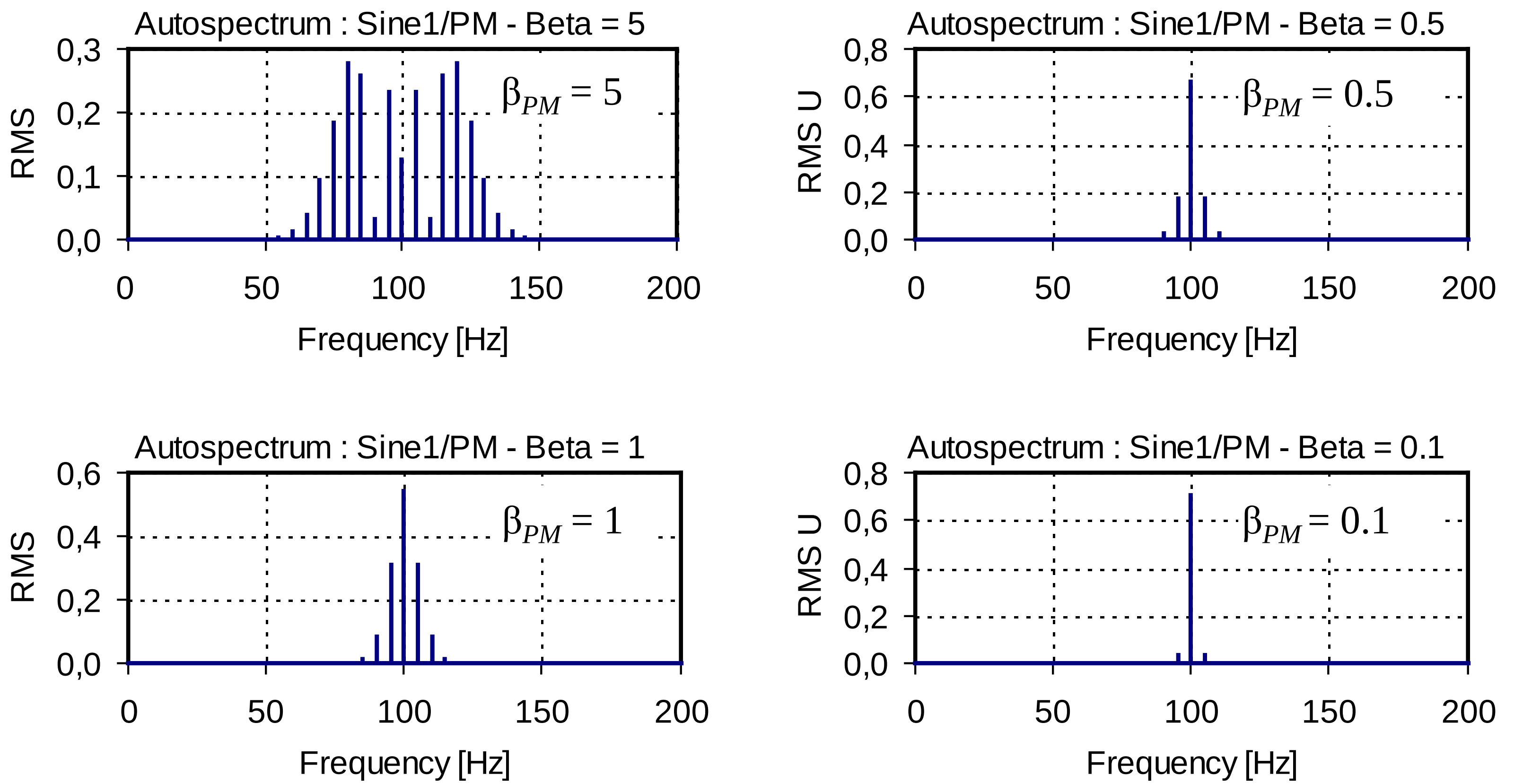


Rotational frequency



Effect of the phase modulation index on the sidebands

The carrying component 100 Hz, amplitude 1, modulation signal frequency 5 Hz

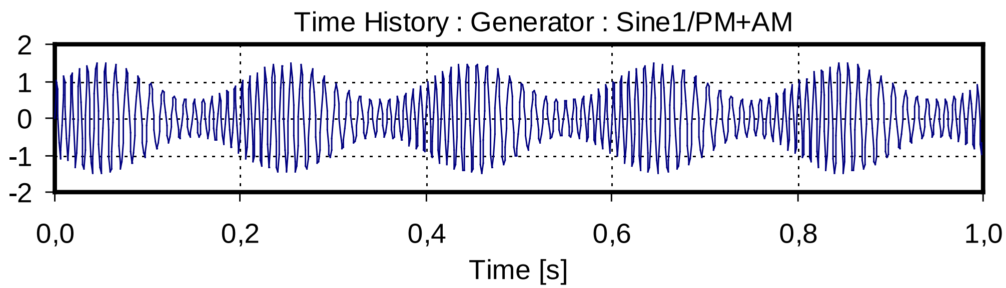


Amplitude and phase modulation I

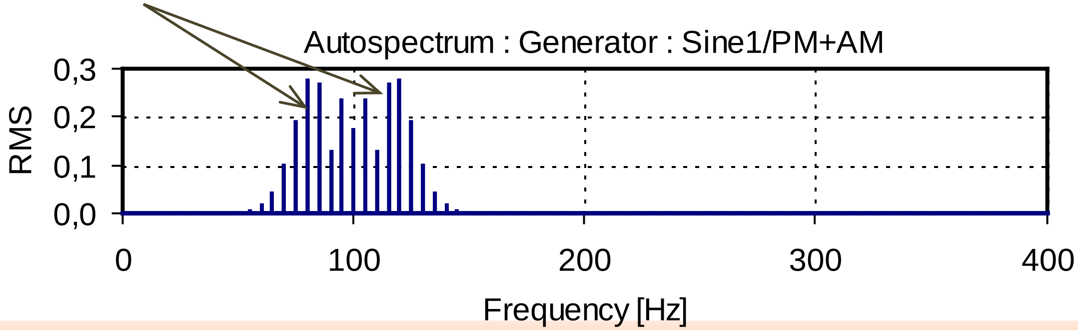
Modulation signals are in phase

Modulated signal　　　Modulation signals　　　Carrying component

$$x(t) = A\left(1 + \beta_{AM}\sin(\omega_M t + \varphi_M)\right)\sin\left(\omega_0 t + \beta_{PM}\sin(\omega_M t + \varphi_M)\right)$$



Symmetric sidebands



β_{AM}, β_{PM} modulation indexes

modulation frequencies

$\omega_M = 2\pi f_M$

carrying frequency

$\omega_0 = 2\pi f_0$

Amplitude and phase modulation II

Modulation signals are out of phase

Modulated signal

Modulation signals

Carrying component

$$x(t) = A\left(1 + \overbrace{\beta_{PM} \sin(\omega_M t + \varphi_M)}\right)\sin\left(\omega_0 t + \overbrace{\beta_{PM} \sin(\omega_M t + \varphi_M + \pi/2)}\right)$$

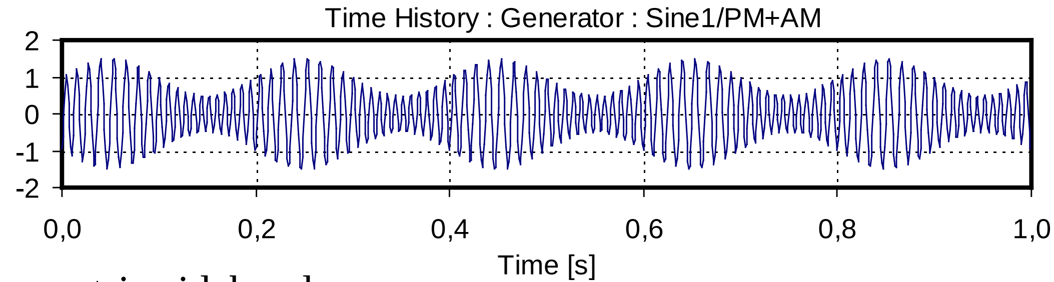
β_{AM}, β_{PM} modulation indexes

modulation frequency

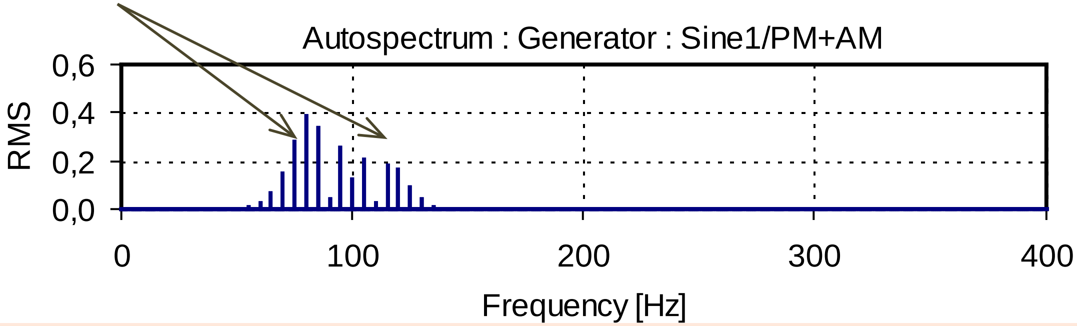
$\omega_0 = 2\pi f_0$

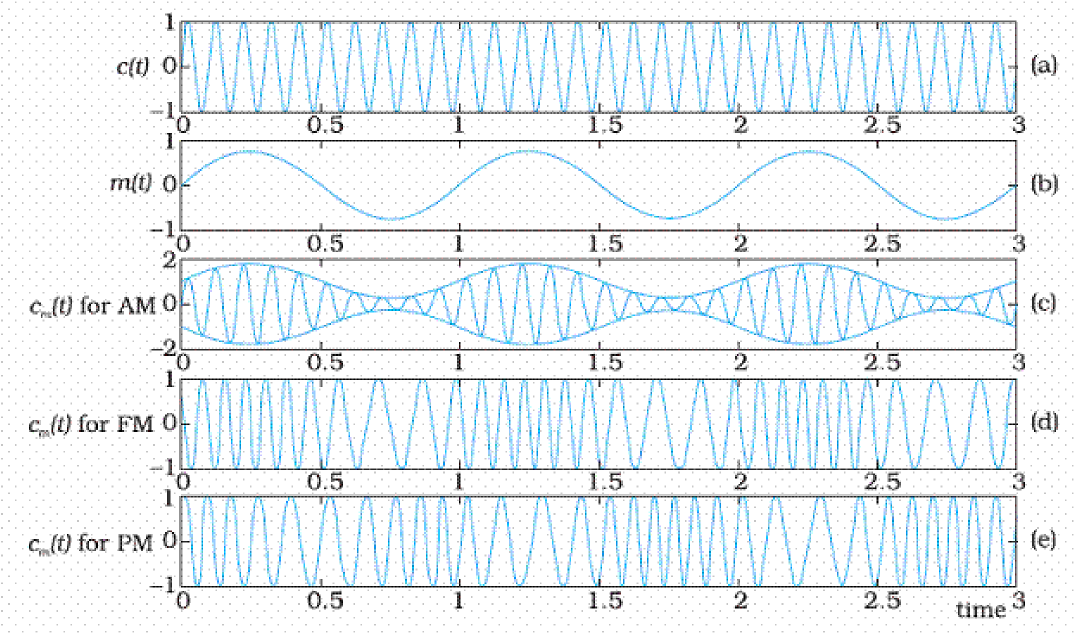
carrying frequency

$\omega_0 = 2\pi f_0$



Non-symmetric sidebands





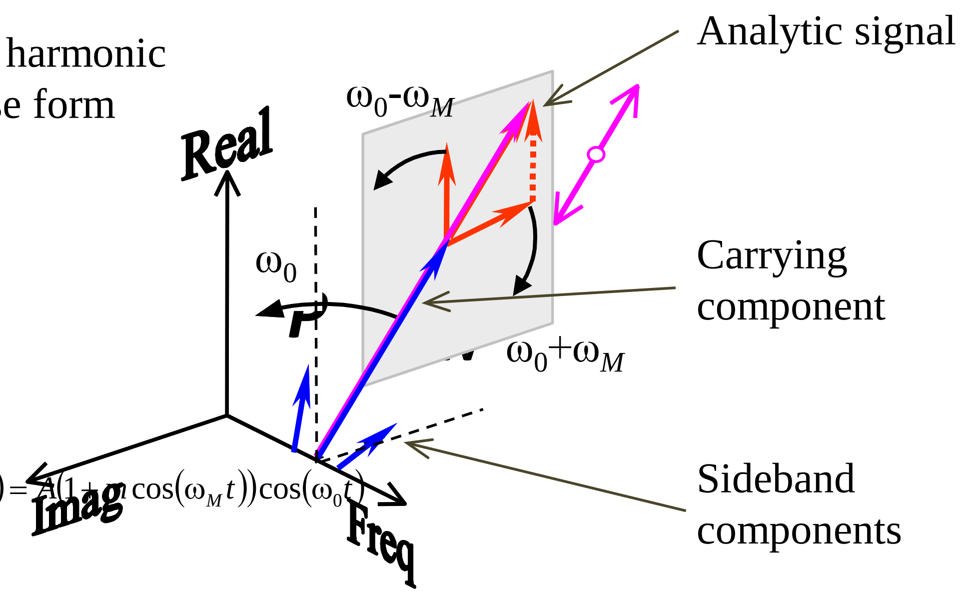
AMPLITUDE AND PHASE DEMODULATION

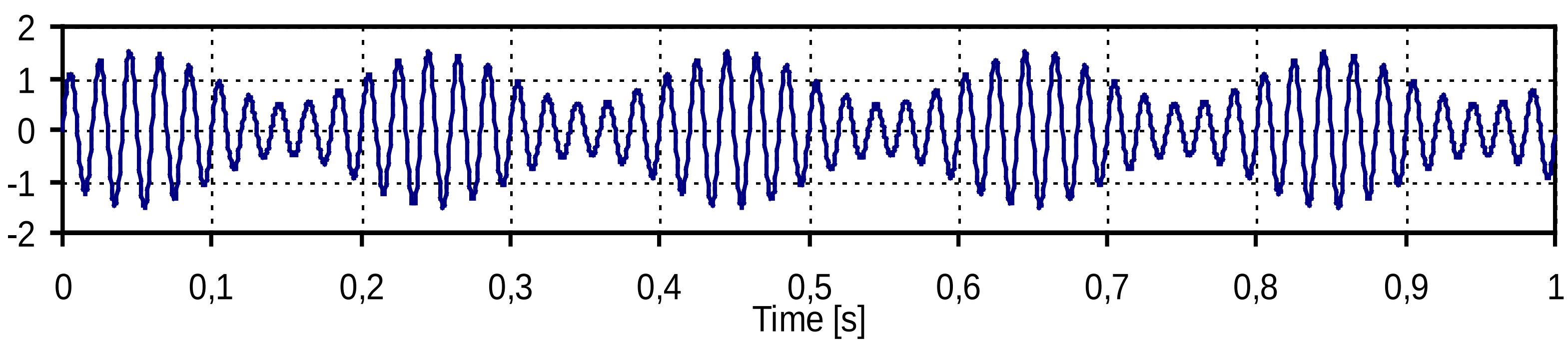
Analytic signals and amplitude modulation

Let $x(t)$ be a real amplitude modulated harmonic signal described by envelope-and-phase form

Modulation signal

$$x(t) = A(1 + m\cos(\omega_M t))\cos(\omega_0 t)$$



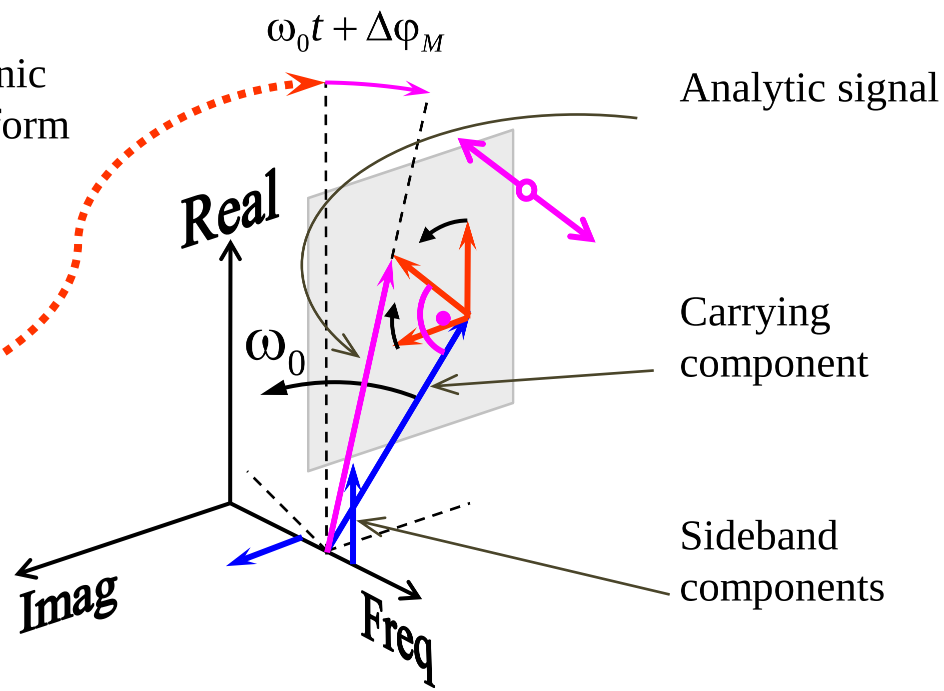


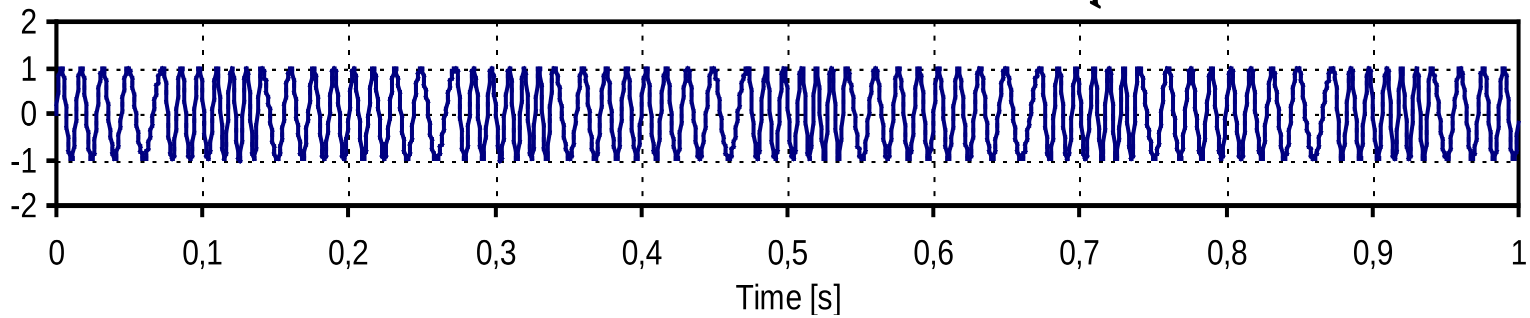
Analytic signals and phase modulation

Let $x(t)$ be a real phase modulated harmonic signal described by envelope-and-phase form

$$x(t) = A\cos\big(\underbrace{\omega_0 t + \Delta\varphi_M}_{\text{Phase}}\big)$$

Modulation signal





Demodulation of the modulated harmonic signal

Firstly the carrier component and its adjacent sidebands have to be filtered using the band pass filter. The output signal is designated by $x(t)$

Secondly the Hilbert transform $y(t)$ of the $x(t)$ signal has to be evaluated using either the FFT transform or the Hilbert transformer to create the analytic signal

$$z(t) = x(t) + j\, y(t)$$

The amplitude modulation signal, referred to as envelope, is as follows

$$A(t) = |z(t)| = \sqrt{x(t)^2 + y(t)^2}$$

The principal value of the phase modulation signal is as follows

$$\varphi_{\mathrm{P.V.}}(t) = \mathrm{Arg}(z(t)) = \arctan(y(t)/x(t))$$

The phase in radians can be computed by the previous formula while taking into the count the value sign of $x(t)$ and $y(t)$. The result will be in the wrapped form which is limiting the angle to the interval $-\pi < \varphi_{\mathrm{P.V.}}(t) \leq +\pi$

To finish the phase demodulation process the wrapped phase has to be unwrapped into

$$\arg(z(t)) = \mathrm{Arg}(z(t)) + 2\pi n(t)$$

where $n(t)$ is a sequence of integer numbers, which depends on time t, for that arg(t) is without discontinuities larger than a permissible value.

The Shannon–Nyquist theorem for sampling of the phase

It is assumed sampling a continuous harmonic signal

$$x(t) = \cos(\omega t) = \cos(2\pi f t)$$

The phase of the mentioned harmonic signal is as follows

$$\varphi(t) = \omega t = 2\pi f t$$

Let the phase difference during the sampling interval be written in the form

$$\Delta\varphi = \varphi_n - \varphi_{n-1} = \omega \Delta t = 2\pi f \Delta t = \frac{2\pi f}{f_S}$$

The Shannon – Nyquist theorem requires

$$2f \le f_S \Rightarrow \frac{f}{f_S} \le \frac{1}{2} \Rightarrow \frac{2\pi f}{f_S} \le \frac{2\pi}{2} = \pi$$

$$\Delta\varphi = \frac{2\pi f}{f_S} \le \pi$$

It can be concluded that the phase change during the sampling interval has to be less then π radians

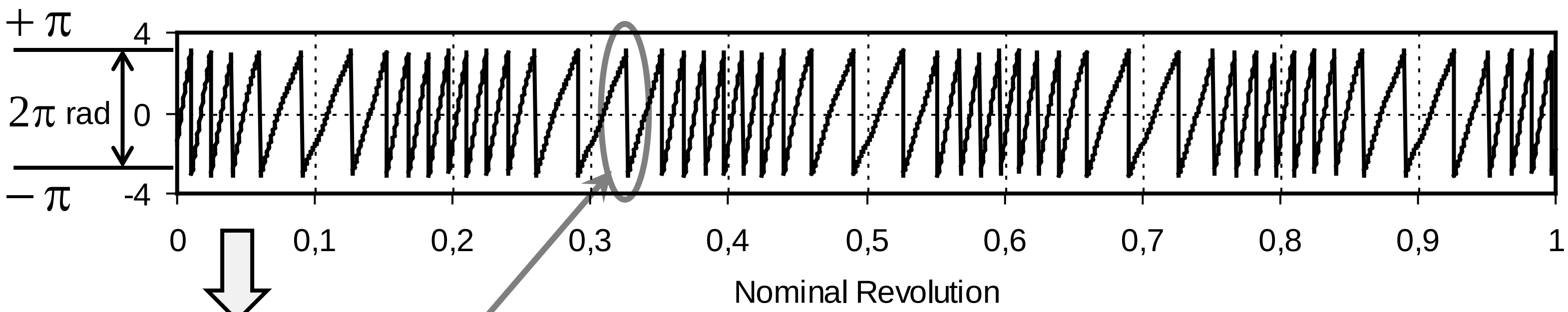
$$|\Delta\varphi| \le \pi$$

This phase change property is basic for unwraping phase signal.

Unwrapping phase and removing the linear trend

Algorithm of the phase unwrapping is based on the phase sampling theorem

The phase demodulation results in in the following signal, which is of the sawtooth wave form.

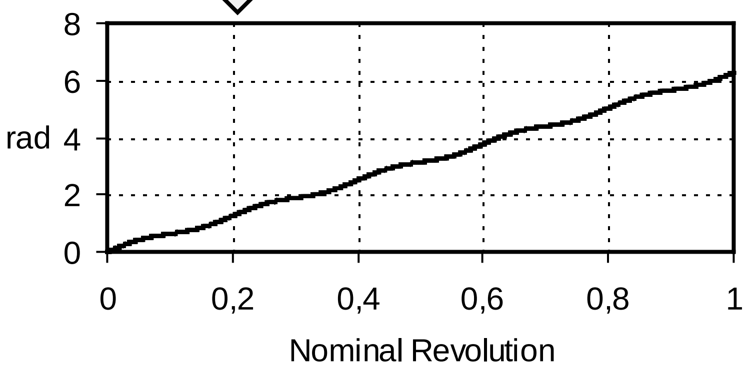


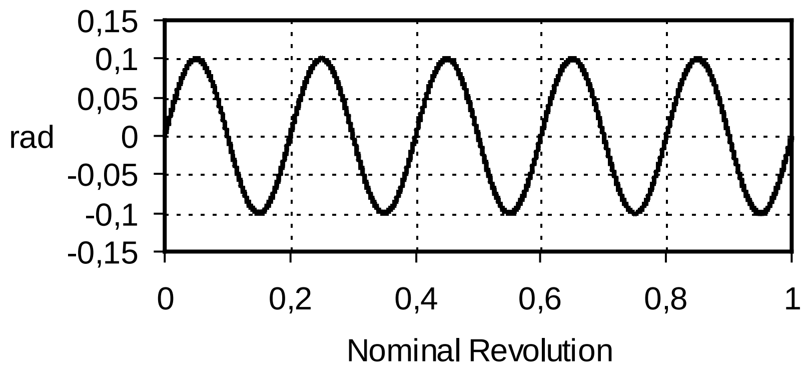
$$\left(2f \le f_{sampl} \Rightarrow |\Delta\varphi| \le \pi\right)$$

Removing discontinuities

Unwrapping algorithm

$$\Delta\varphi < -\pi \Rightarrow \varphi + 2\pi \to \varphi,\ \Delta\varphi > +\pi \Rightarrow \varphi - 2\pi \to \varphi$$





An alternative procedure for computing instantaneous frequency

It is not always necessary to calculate the unwrapped phase. To calculate the instantaneous frequency of the modulated harmonic signal it is possible to use the following formulas

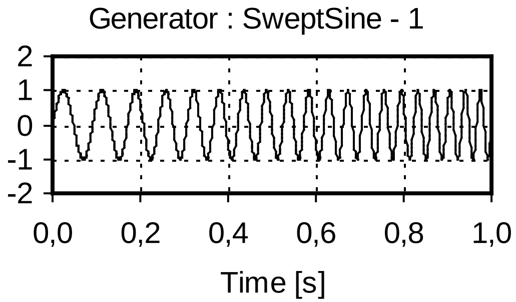
Phase …………….... $\varphi(t) = \arctan\left(\dfrac{y(t)}{x(t)}\right)$

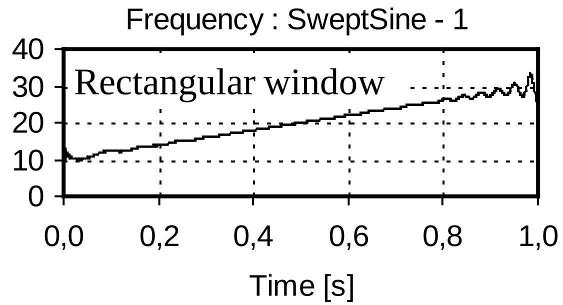
Envelope …………. $e(t) = \sqrt{x^2(t) + y^2(t)}$

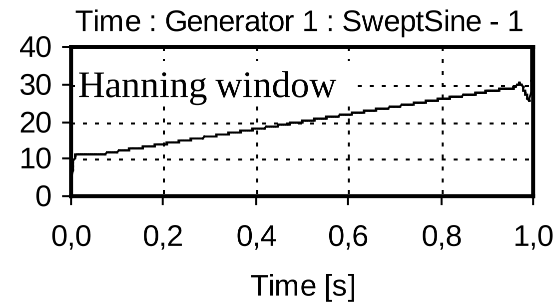
Angular frequency … $\omega(t) = \dfrac{d\varphi(t)}{dt} = \dfrac{d\left(\arctan\left(\dfrac{y(t)}{x(t)}\right)\right)}{dt} = \dfrac{\dfrac{dx(t)}{dt}y(t) - x(t)\dfrac{dy(t)}{dt}}{x^2(t) + y^2(t)}$

Phase ……………… $\varphi(t) = \int_0^t \omega(\tau)\, d\tau$

Let the frequency of the swept sine signal be running-up from 10 to 30 Hz



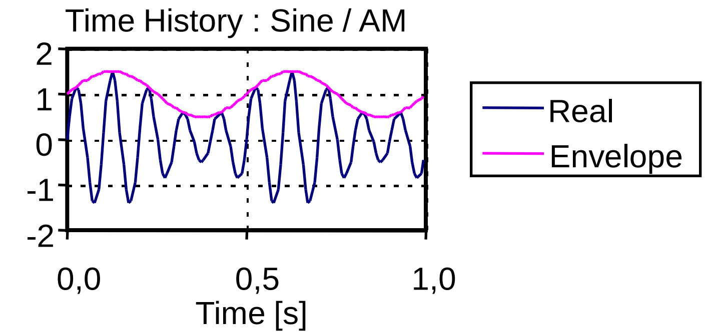


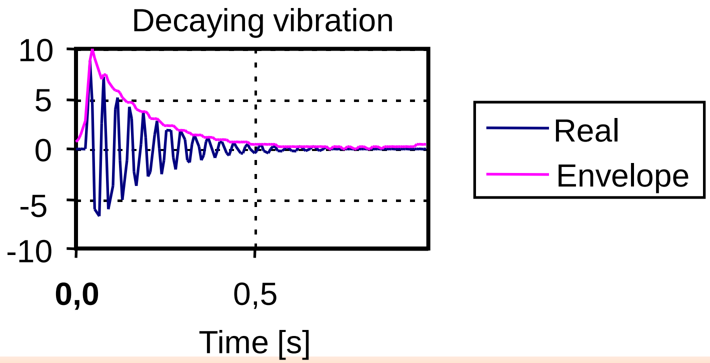


Envelope analysis

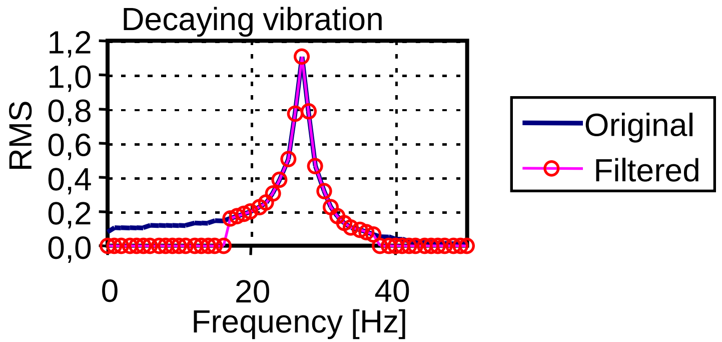
The amplitude demodulation is referred to as envelope analysis. The examples shows computing the envelope for a broadband signal or signal zoomed around a resonance frequency with the use of the bandpass filter.

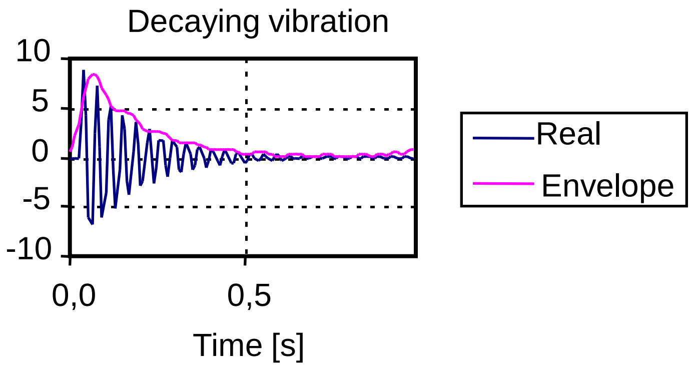
The envelope is computed for the full frequency spectrum



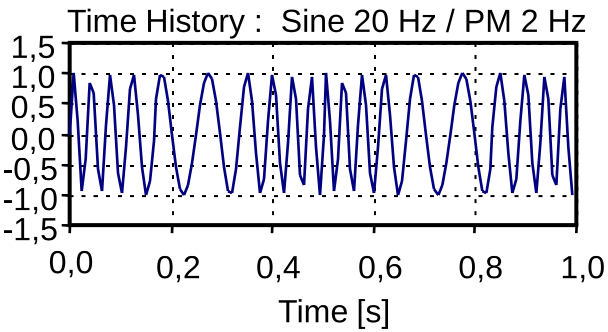


The envelope is computed for narrow band part of frequency spectrum



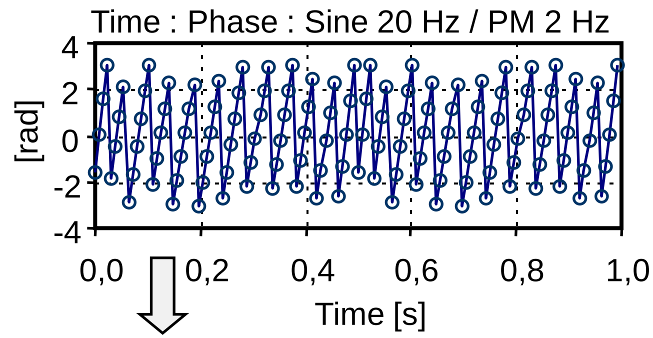


Phase demodulation

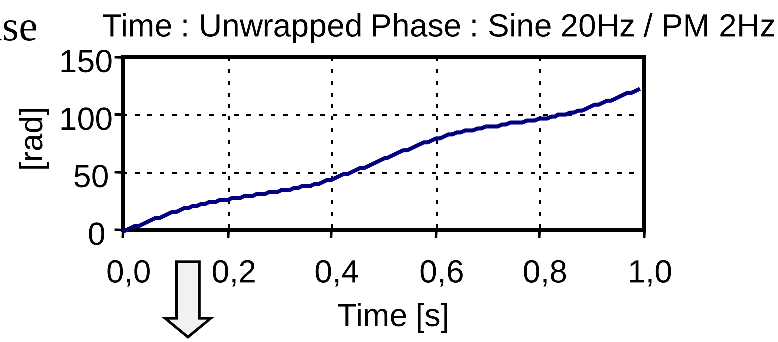


Phase modulated signal

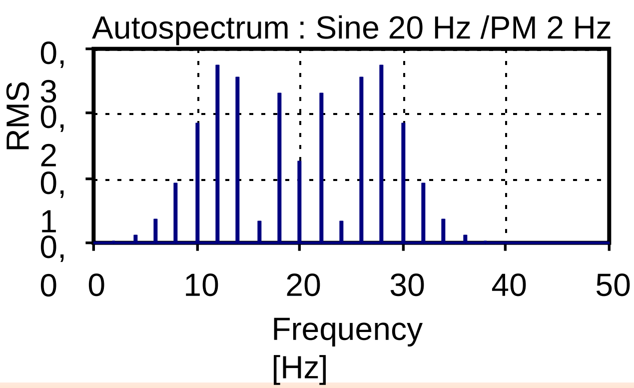
Wrapped phase (principal values of phase])



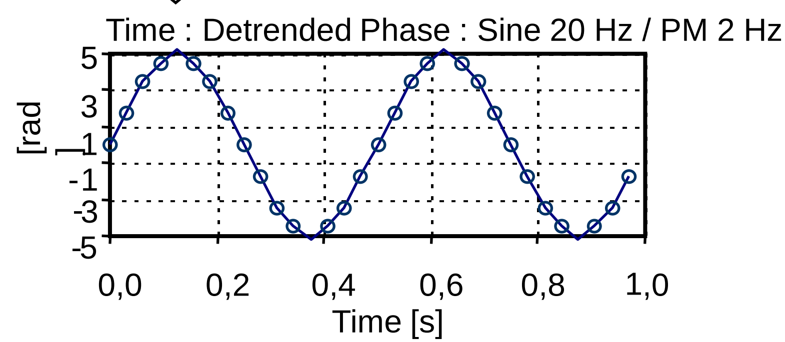
Unwrapped phase



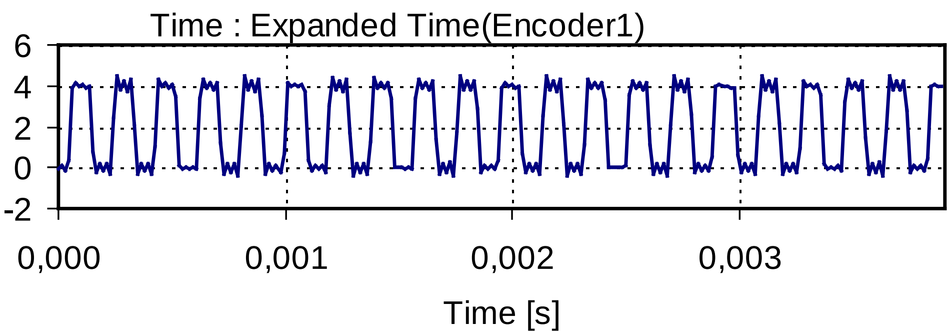
Frequency spectrum of the modulated signal

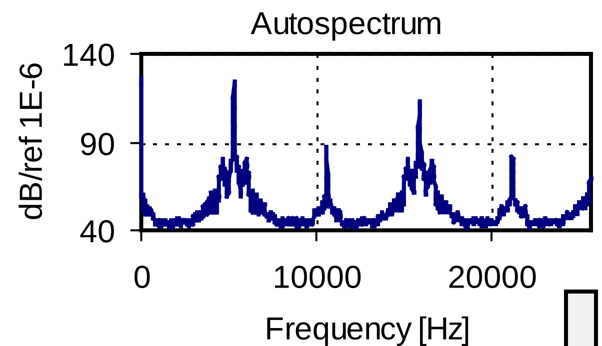


Detrend phase

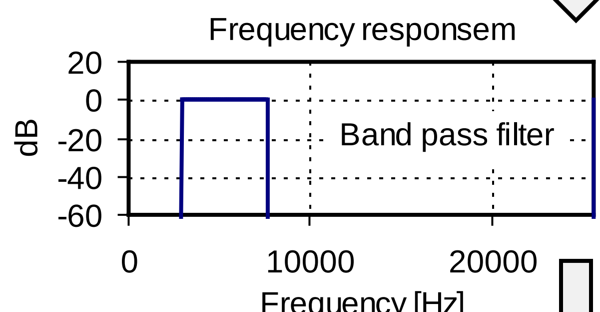


Conversion of the frequency modulated impulse signal to the harmonic signal



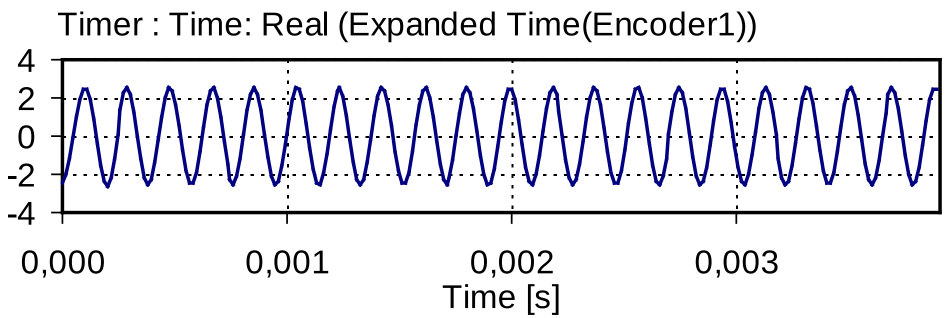


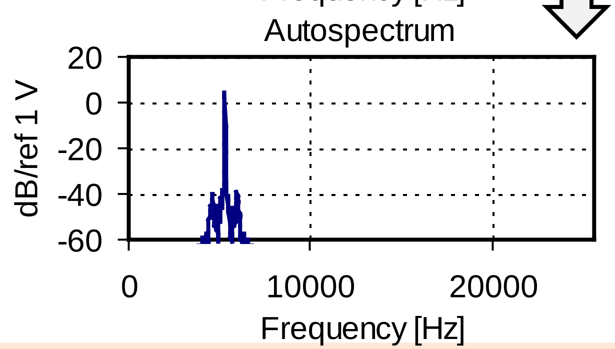
The original impulse signal



Band pass filtration

The resulting harmonic signal





References I

[Shie Qian] Shie Qian Introduction to Time-Frequency Wavelet Transform, Prentice Hall PTR, Upper Saddle River, New Jersey 07458, 2002

[Oppenheim & Schafer] Oppenheim, A.V. & Schafer, R.W. Discrete time signal processing. Prentice Hall PTR, Upper Saddle River, New Jersey 07458, 1989.

[Vold & Leuridan] Vold, H., Leuridan, J., "High Resolution Order Tracking at Extreme Slew Rates, using Kalman Tracking Filters", SAE Technical Paper No. 931288, Noise & Vibration Conference & Exposition, Traverse City, Michigan, May, 1993.

[Mitra] Mitra, S.K. Digital signal processing. McGraw-Hill, Avenue of the Americas, New York, NY, 10020, 2001.

[Orfanidis] Orfanidis, S.J. Optimum signal processing. Macmillan Publishing Company, New York1988.

[Haykin] Haykin, S. Adaptive filter theory. Prentice Hall PTR, Upper Saddle River, New Jersey 07458, 1996.

[Tůma, 1997] Tůma, J. Zpracování signálů získaných z mechanických systémů. 1. vyd. Praha : Sdělovací technika, 1997. 174 s. ISBN 80-901936-1-7

[Crocker] Crocker, M. Handbook of noise and vibration control. New York: Wiley, 2007.

[Feldbauer & Holdrich] Feldbauer, Ch. & Holdrich, R. Realisation of a Vold-Kalman Tracking Filter – A Least Square Problem. In Proceedings of the COST G-6 Conference on Digital Audio Effects (DAFX-000), Verona Italy, December 7-9, 2000.

[Åstrőm] Åstrőm, K. J. Introduction to Stochastic Control Theory. Dover Publication, Inc. Mineola, New York 1970.

References II

[Tůma,1998] Tůma, J. Složité systémy řízení, I. Díl: Regulace soustav s náhodnými poruchami, 1. vyd. Ostrava : Skripta VŠB - TU Ostrava, 1998. 151 s. ISBN 80-7078 - 534 - 9. (in Czech)

[Tůma,2005] Tůma, J. Setting the passband width in the Vold-Kalman order tracking filter. In: Twelfth International Congress on Sound and Vibration, (ICSV12). Lisabon, July 11-14, 2005, Paper 719.

[Tůma,2005] Tůma, J. The passband Width of the Vold-Kalman Order Tracking Filter. Sborník vědeckých prací VŠB-TU Ostrava, řada strojní r. LI, 2005. č. 2, příspěvek č. 1485, s. 149-154.

[Tůma,2009] Tůma, J. Diagnostika strojů. 1. vyd. Skripta VŠB-TU Ostrava, 2009.

[Randall] Randall,B. Frequency analysis, Brüel & Kjær, Revision September 1987.

[Josef Ström Bartůněk] Fundamental Image Processing and 2-D Fourier Transform. Lectures done at VSB – TU Ostrava.

[Goertzel] Goertzel, G. (1958). An Algorithm for the Evaluation of Finite Trigonomentric Series. The American Mathematical Monthly.

[Welch & Bishop] An Introduction to the Kalman Filter by Greg Welch and Gary Bishop, Dept of Computer Science University of North Carolina at Chapel Hill Chapel Hill, NC 27599-3175.

[Roumeliotis & Sukhatme & Bekey] Roumeliotis, Sukhatme, Bekey : Circumventing Dynamic Modeling: Evaluation of the Error-State Kalman filter applied to Mobile Robot Localization

[Donadio] Donadio, M. (2000) CIC Filter Introduction "Hogenauer introduced an important class of digital filters called 'Cascaded Integrator-Comb', or 'CIC' for short (also sometimes called 'Hogenauer filters').

References III

http://www.mathworks.com/access/helpdesk/help/toolbox/wavelet/

http://users.rowan.edu/~polikar/WAVELETS/WTtutorial.html

http://www.it.uu.se/edu/course/homepage/bild1/vt05/Lectures/L12VT05.pdf

http://www.qi.tnw.tudelft.nl/~lucas/education/tn254/2002/Fourier%20transform%20of%20a%20Gaussian.pdf

http://jenshee.dk/signalprocessing/signalprocessing.html

http://www.cs.unc.edu/~welch/kalman/#Anchor-49575