EVROPSKÁ UNIE
MINISTERSTVO ŠKOLSTVÍ, MLÁDEŽE A TĚLOVÝCHOVY
OP Vzdělávání pro konkurenceschopnost
VYSOKÉ UČENÍ TECHNICKÉ V BRNĚ

INVESTICE DO ROZVOJE VZDĚLÁVÁNÍ

Mathematical introduction to chaos theory

RNDr. Petra Augustová, Ph.D. (ÚTIA, AV ČR, v. v. i.)

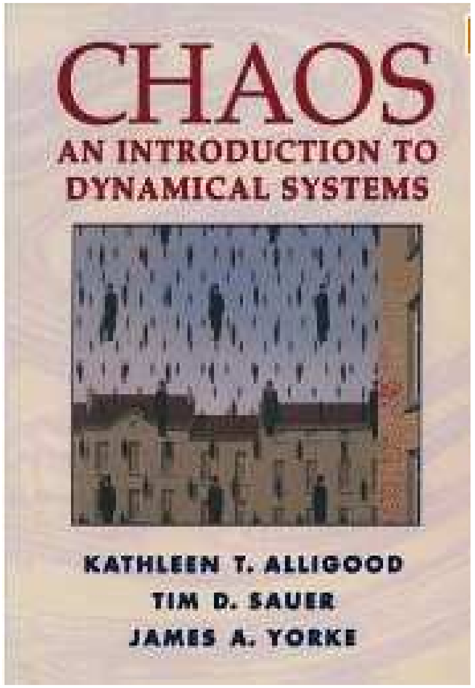
2. listopadu 2012

Tato prezentace je spolufinancována Evropským sociálním fondem a státním rozpočtem České republiky.

Introduction

The main source for this text is the following book:

- K. T. Alligood, T. D. Sauer and J. A. Yorke: Chaos – an introduction to dynamical systems, Springer, 2000.



History

The founders of dynamical systems:

- Sir Isaac Newton (1643–1727)
- James Clerk Maxwell (1831–1879)
- Henri Poicaré (1854–1912)
- Computers!

One–dimensional maps

A *dynamical system* is a set of possible states (also called phase space or state space), together with a law of evolution (rule) that determines the present state in terms of past states. This process is deterministic.

We will consider here discrete dynamical systems, where the rule is applied at discrete times (for example n-dimensional maps).

Iterations

By discrete dynamical system we mean a one-dimensional map (function with domain=range) and its iterations.

The n–th iterate of f is the map $f^n = f \circ f^{n-1}$, $n \in \mathbb{N}$.

The negative iterates are given by $f^{-n} = (f^n)^{-1}$, $n \in \mathbb{N}$.

We use the notation $f^0 = f$.

The *(forward) orbit* of a point x is the set $\{f^n(x) \mid n \in \mathbb{N}\}$ where x is called the *initial value*.

A point x is called a *fixed point* of a map f if it satisfies $f(x) = x$.

Example: population growth model 1

Linear map $f(x) = 2x$

x denotes the population (in millions)

$x_n = f(x_{n-1}) = 2x_{n-1}$, where n is time and x_n the population at time n

Nonzero population grows to infinity (exponential growth).

Such model is not real for too long.

The fixed point of f is 0.

Example: population growth model 2

Nonlinear map $g(x) = 2x(1 - x)$

Compare the models for small and large x! Use a calculator, take the starting point as $x = 0.01$ and discuss the outcome.

This model gives the same saturation with different starting points.

What are the fixed points of g?

Graphical representation of an orbit.

cobweb plot as a rough sketch of an orbit

Observations:

fixed points are intersections with the diagonal

If the graph is above the diagonal the orbit moves to the right, if below then to the left

Stability

Definition. Let p be a fixed point of a real map. If all points sufficiently close to p are attracted to p, then p is called a *sink* or *attracting fixed point*. If all points sufficiently close to p are repelled from p, then p is called a *source* or a *repelling fixed point*.

Stability test 1

Theorem. Let f be a smooth map on $\mathbb{R}$ wit a fixed point p.

a) If $|f'(p)| < 1$, then p is a sink.

b) If $|f'(p)| > 1$, then p is a source.

Check in the logistic model 2 that $x = 1/2$ is a sink and $x = 0$ is a source of g.

Note that for $f(x) = ax$ with $|a| < 1$, the sink at 0 attracts everything!

Periodic points

Definition. Let f be a real map. We call p a *periodic point of period k* (shortly *period-k point*) if $f^k(p) = p$ and k is smallest such positive integer. The orbit of p is called a *periodic orbit of period k* (shortly *period-k orbit*). The period-k point p is a *periodic sink (source)* if p is a sink (source) for the map f^k.

Example: logistic family of maps $g_a(x) = ax(1-x)$

$a = 3.3$ two repelling fixed points and most orbits "attracted"to a couple of points

Stability test 2

Stability test for periodic orbits.

The k-periodic orbit $\{p_1, \ldots, p_k\}$ is a sink if

$$|f'(p_k) \ldots f'(p_1)| < 1$$

and a source if

$$|f'(p_k) \ldots f'(p_1)| > 1.$$

The logistic family

$g_a(x) = ax(1-x)$ for different values of a

When $0 \leq a < 1$, there is a sink at $x = 0$ and every initial point between 0 and 1 is attracted to this sink (i.e. small populations with small reproduction rate die out).

When $1 < a < 3$, there is a sink at $x = (a-1)/a$ (check the derivative!), i.e. small populations grow to a stable state.

The logistic family

When $a > 3$, the fixed point $x = (a-1)/a$ is unstable (check the derivative!), and a period-2 sink appears for $a = 3.3$. When a grows further (approx. 3.45), this period-two sink becomes unstable. For even slightly larger values of a, the situations becomes much more complicated! There are many new periodic points!

When $a > 4$, there are no attracting sets.

Bifurcation diagram

a) Choose a value of a, start with $a = 1$.

b) Choose $x \in [0, 1]$ and calculate its orbit.

c) Ignore the first 100 iterates and plot the iterates starting with 101.

Note 1: The point x was chosen randomly, but nothing changes to the diagram if we pick a different point.

Note 2: For larger a there are many unstable periodic orbits that we do not see in the bifurcation diagram.

Iterations

Special case $G(x) = 4x(1-x)$.

Since this map has no sinks, where do the orbit go?

Draw graphs of G^2 and G^3.

Simple analysis shows that G^k has 2^k fixed points in the unit interval. It is not difficult to show that the map G has a period-k orbit for any integer k.

Hence G has infinitely many periodic orbits.

Question. Is this chaos?

Sensitive dependence on initial conditions

Definition. Let f be a map on $\mathbb{R}$. A point $x \in \mathbb{R}$ has *sensitive dependance on initial conditions* if there is a nonzero number d (we can call it distance) such that some points arbitrarily close to x are eventually mapped at least a distance d from the corresponding iteration of x. Precisely, there is a number $d > 0$ such that any neighborhood U of x contains a point y such that $|f^k(y) - f^k(x)| \geq d$ for some nonnegative integer k. We call the point x a *sensitive point*.

Example

$f(x) = 3x \mod 1$ on the unit interval

discontinuous map but the discontinuity is not an interesting problem here: imagine the map as a map on the circle of circumference 1 and the map is continuous here

Definition. We call a point *eventually periodic* with period p for the map f if for some positive integer N, $f^{n+p} = f^n(x)$ for all $n \geq N$ and p is the smallest such number.

Example

Note: A point x is eventually periodic for f if and only if it is a rational number.

f has the main property of chaos, the sensitive dependance on initial conditions: Taking two points close to each other, they will move apart under iteration.

Itineraries

The idea is to use coding (discrete symbols) for orbits that keeps informations.

For the logistic map G, use the symbol **L** to the left subinterval $[0, 1/2]$ and the symbol **R** to the right subinterval $[1/2, 1]$.

Start with the symbol according to the position of x and continue depending on the position of the following iterates.

Transition graph

We can also use a transition graph.

It is a fully connected graph for G, i.e. all possible sequences of **L** and **R** are possible.

We always find a neighbor to x that eventually moves apart by a distance at least $d = 1/4$. Just find the subintervals ...LR and ...RL. That means the sensitive dependence on initial conditions.

Computers!!!

Period three

A continuous map that has an orbit of periods three turns to be complicated in the following sense:

1) the map has periodic orbits of all periods, i.e. all possible integers (we state the Sharkovsky theorem later);

2) there is a large set of sensitive points, actually an uncountable set (proved by Li&Yorke in 1975).

Conclusion. Period three implies chaos!!!

Two–Dimensional Maps

- n-body problem
- Poincaré map
- quadratic map $f(x, y) = (a - x^2 + by, x)$.

Fixed points

Definition. Let f be a map on $\mathbb{R}^m$ and p be a fixed point of f. If there is an $\varepsilon > 0$ such that for all x in the ε-neighborhood $N_\varepsilon(p)$, $\lim_{k\to\infty} f^k(x) = p$, then p is a *sink* or *attracting fixed point*. If there is an ε-neighborhood $N_\varepsilon(p)$, such that each $x \in N_\varepsilon(p) \setminus \{p\}$ eventually maps outside of $N_\varepsilon(p)$, then p is a *source* or *repeller*.

We call a fixed point a *saddle* if it has some attracting direction and some repelling direction.

Linear maps

Definition. A map A from $\mathbb{R}^m$ to $\mathbb{R}^m$ is *linear* if for each $a, b \in \mathbb{R}$ and $u, v \in \mathbb{R}^m$, $A(au + bv) = aA(u) + bA(v)$. It can be represented as a multiplication by an $m \times m$ matrix.

Theorem. Let A be a linear map on $\mathbb{R}^m$ represented by the matrix A. Then

1. The origin is a sink if all eigenvalues of A are smaller than 1 in absolute value;
2. The origin is a source if all eigenvalues of A are larger than 1 in absolute value.

Hyperbolicity

Definition. Let A be a linear map on $\mathbb{R}^m$. We say that A is *hyperbolic* if A has no eigenvalues of absolute value one. If a hyperbolic map A has at least one eigenvalue of absolute value greater than one and at least one eigenvalue of absolute value smaller than one, then the origin is called a *saddle*.

Nonlinear maps

Definition. Let $f = (f_1, f_2, \ldots, f_m)$ be a map on $\mathbb{R}^m$ and let $p \in \mathbb{R}^m$. The *Jacobian matrix* of f at p is the matrix

$$Df(p) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(p) & \cdots & \frac{\partial f_1}{\partial x_m}(p) \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1}(p) & \cdots & \frac{\partial f_m}{\partial x_m}(p) \end{pmatrix}$$

of partial derivatives evaluated at p.

Stability test

Theorem. Let f be a map on $\mathbb{R}^m$ and p its fixed point.

1. If the magnitude of each eigenvalues of $Df(p)$ is less than 1, then p is a sink;
2. If the magnitude of each eigenvalues of $Df(p)$ is greater than 1, then p is a source.

Hyperbolicity

Definition. Let f be a map on $\mathbb{R}^m$ and p its fixed point. Then this fixed point is *hyperbolic* if none of the eigenvalues of $Df(p)$ has magnitude one. If p is hyperbolic and at least one eigenvalue of $Df(p)$ has magnitude greater than one and a least one eigenvalue has magnitude less than one, then p is called a *saddle*. (For a periodic point of period k, replace f by f^k.)

Observations

Saddles are unstable!

Calculate the Jacobian matrix for the Hénon map with $a = 0, b = 0.4$. Notice that it has two fixed points $(0, 0)$ and $(-0.6, -0.6)$. What is their nature?

Bifurcation diagram for the Hénon map with $b = 0.4$ and $0 \leq a \leq 1.25$ (horizontal axis).

Stable and unstable manifolds

A saddle fixed point is unstable. But there are points close to the saddle that do not move away. We will call the points that converge to the saddle the *stable manifold*.

Definition. Let f be a smooth one-to-one map on $\mathbb{R}^2$ and let p be a saddle fixed point or a saddle periodic point of f. The *stable manifold* of p, denoted $S(p)$, is the set of points x such that $|f^n(x) - f^n(p)| \to 0$ as $n \to \infty$. The *unstable manifold* of p, denoted $U(p)$, is the set of points x such that $|f^{-n}(x) - f^{-n}(p)| \to 0$ as $n \to \infty$.

Example

Linear map $f(x, y) = (2x, y/2)$.

Origin is a saddle with points in the y-axis converging to this saddle (all other diverge to infinity):

The eigenvectors are (1,0) corresponding to the stretching by eigenvalue 2 and (0,1) corresponding to the shrinking by eigenvalue 1/2 (this is the stable manifold of 0).

We can also describe the unstable manifold as the stable manifold of the inverse map of f.

Notes 1

For linear maps, the stable and unstable manifolds of saddles are linear subspaces. For nonlinear maps stable and unstable manifolds cannot be found directly and we need to use computer to approximate (Hénon).

The stable and unstable manifolds in the plane are one-dimensional sets (lines, curves). They have a big effect on the dynamics and chaos of the map.

A stable manifold cannot cross itself nor a different stable manifold. But a stable manifold can cross the unstable manifold of the same fixed point in homoclinic point (Poicaré). Such intersection necessarily means infinitely many such intersections.

Notes 2

The stable and unstable manifolds are invariant sets. Realize the complexity of the dynamics near the homoclinic points and the existence of sensitive dependence on initial conditions.

As for the one-dimensional mod-map, we can have a two-dimensional map with infinitely many periods (e.g. defined on a torus).

Question. Is the solar system stable?

Chaos

Roughly a chaotic orbit:

keeps the unstable behavior (not fixed, not periodic, not eventually attracted to sink).

arbitrarily close to any its point there are point that move away.

Separation rate

The stability of fixed and periodic points depends on derivative.

For periodic point of period k, we calculate the derivative of the k-th iterate as the product of derivatives at the k points of the orbit (chain rule). If this product is $a > 1$, it means the separation of close point is approximately a per k iterates. So we can think about the average separation rate per iterate. It is easy to see it is $a^{1/k}$ in this case.

We want to have this concept also for points that are not fixed nor periodic.

Lyapunov exponents

Definition. Let f be a smooth map on $\mathbb{R}$. The *Lyapunov number* $L(x_1)$ of the orbit $x_1, x_2, \ldots$ is defined as

$$L(x_1) = \lim_{n\to\infty} (|f'(x_1)| \ldots |f'(x_n)|)^{\frac{1}{n}},$$

if the limit exists. The *Lyapunov exponent* $h(x_1)$ is defined as

$$h(x_1) = \lim_{n\to\infty} (1/n)(\ln|f'(x_1)| + \cdots + \ln|f'(x_n)|),$$

if the limit exists.

Note that h exists if and only if L exists and is nonzero, and $\ln L = h$.

Asymptotic stability

Definition. Let f be a smooth map. An orbit $x_1, x_2, \ldots$ is called *asymptotically periodic* if it converges to a periodic orbit. It means that there exists a periodic orbit $y_1, y_2, \ldots y_k$ such that

$$\lim_{n\to\infty} |x_n - y_n| = 0.$$

An orbit attracted to a sink is asymptotically periodic. The extreme case of the asymptotically periodic is eventually periodic.

Theorem. Let f be a map on $\mathbb{R}$. If the orbit $x_1, x_2, \ldots$ of f satisfies $f'(x_i) \neq 0$ for all i and is asymptotically periodic to a periodic orbit, then the two orbits have the same Lyapunov exponents (if they exist).

Chaotic orbits

The interesting case is if the orbit is not asymptotically periodic.

Definition. Let f be a map on $\mathbb{R}$. Let $x_1, x_2, \ldots$ be a bounded orbit of f. This orbit is chaotic if

1. it is not asymptotically periodic;
2. the Lyapunov exponent $h(x_1) > 0$.

Binary expansion

The binary expansion of a number x in the unit form is in the form $.a_1a_2\ldots$ with each a_i beeing the 2^{-i} contribution to x.

The process goes as follows:

multiply the number by 2 and take the integer part as the bit a_1

repeat the process...

Note: this is the map $f(x) = 2x(\mod 1)$

Simple example

$f(x) = 2x(\mod 1)$ on the real line

It has positive Lyapunov exponents and chaotic orbits.

Since the map is not continuous, we consider only points that do not map to the point of discontinuity at $1/2$.

$$h(x_i) = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} \ln|f'(x_i)| = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln 2 = \ln 2.$$

Each orbit that does not fall in $1/2$ and is not asymptotically periodic is chaotic!

Shift

In binary expansion it cuts the left bit:

$$1/5 = .0011\overline{0011}$$

$$f(1/5) = .011\overline{0011}$$

$$f^2(1/5) = .11\overline{0011}$$

$$f^3(1/5) = .1\overline{0011}$$

$$f^4(1/5) = .\overline{0011} = 1/5$$

Periodic point (repeating expansion), eventually periodic (=asymptotically periodic). Hence chaotic points have eventually periodic binary expansion (not rational).

Tent map.

$T(x) = 2x$ if $x \leq 1/2$ and $T(x) = 2(1-x)$ if $x \geq 1/2$.

Similar shape as the logistic map (but not smooth).

We can also use the itineraries of orbits.

T has infinitely many chaotic orbits (in fact, the absolute value of the slope is 2 except at the peak at $1/2$, therefore the Lyapunov exponent of an orbit is $\ln 2$ if it exists).

Conjugacy

Definition. The maps f and g are *conjugate* if there is a continuous one–to–one change of coordinates, i.e. if $C \circ f = g \circ C$ for a continuous one–to–one map C. This map is called the *conjugacy*.

Check that $C(x) = (1 - \cos \pi x)/2$ is a one–to–one continuous map that conjugates T and G!

Theorem. Let f and g be conjugate maps and C the conjugacy. If x is a period-k point of f, then $C(x)$ is a period-k point of g. If moreover C' is never zero on the periodic orbit of f, then

$$(g^k)'(C(x)) = (f^k)'(x).$$

Consequences

All periodic points of the logistic map G are sources.

The logistic map G has chaotic orbits.

Note: Unfortunately for most parameter values a in the logistic family no useful conjugacy exists...

Sharkovskii's Theorem

the Sharkovskii's ordering is:

$$3 \prec 5 \prec 7 \prec 9 \prec \cdots \prec 2{\cdot}3 \prec 2{\cdot}5 \prec \cdots \prec 2^2{\cdot}3 \prec 2^2{\cdot}5 \prec \cdots$$

$$\cdots \prec 2^3{\cdot}3 \prec 2^3{\cdot}5 \prec \cdots \prec 2^4{\cdot}3 \prec 2^4{\cdot}5 \prec \cdots \prec 2^3 \prec 2^2 \prec 2 \prec 1.$$

Sharkovskii's Theorem. Assume that f is a continuous map on the unit interval and has a period-k orbit. If $p \prec q$, then f has a period-q orbit.

Fractals

Term "fractal"first used in the 1960's by B. Mandelbrot (mathematician at IBM).

No common definition but fractal has some of the following properties:

Complicated structure (while scaling)
Repetition of structures (self-similarity)
Fractal dimension not an integer

Cantor set

Simplest fractal construction:

- We start with the unit interval $[0,1]$.
- Remove the open interval $(1/3, 2/3)$ i.e. the middle-third of I. We call the remaining set $C_1 = [0, 1/3] \cup [2/3, 1]$.
- Remove the middle-thirds of the remaining intervals in C_1, i.e. $(1/9, 2/9) \cup (7/9, 8/9)$. The remaining set $C_2 = [0, 1/9] \cup [2/9, 3/9] \cup [6/9, 7/9] \cup [8/9, 1]$
- Repeat this process of removing the middle-third open intervals. The limiting set of this precess is called the *middle-third Cantor set* and is denoted by $C = C_\infty$.

What is its length? Size? Measure? Etc.

Theorem. The middle-third Cantor set C consists of all numbers in the unit interval that can be expressed in base 3 using only the digits 0 and 2.

The number $.\overline{02} = 1/4$ is in C.

There is a one-to-one correspondence between the unit interval and C. Hence, C is an uncountable set!
Cantor set is self-similar (magnifying a part of the set gives again the entire Cantor set).
Other methods producing the Cantor set (probabilistic processes, as attractors).
In dimension two, the most famous Cantor sets are the Sierpinski gasket and and the Sierpinski carpet.

Fractals produced by maps

Tent map with slope $a > 0$
$T_a(x) = ax$ if $x \leq 1/2$ and
$T_a(x) = a(1-x)$ if $x \geq 1/2$

For $a = 3$, basin of infinity (points that diverge to $-\infty$) is:

- intervals $(-\infty, 0)$ and $(1, +\infty)$ converge to $-\infty$.
- interval $(1/3, 2/3)$ converge to $-\infty$ (since it is mapped to $(1, +\infty)$).
- further we find the intervals $(1/9, 2/9)$ and also $(7/9, 8/9)$ (mapped to the previous interval)
- and so on...

Mandelbrot set

complex quadratic map $P_c(z) = z^2 + c$

The orbit of 0 has an important role here (for some parameter c is is a fixed point, for some not...).

We define the *Mandelbrot set* by

$$M = \{c : 0 \text{ is not in the basin of infinity for the map } P_c\}.$$

Julia sets

The boundary of the basin of infinity is called the *Julia set*. It is defined equivalently (for polynomials) as the set of repelling fixed and periodic points together with the limit points of this set.

Use computer to draw some Julia sets!

Fractals dimension

For an interval J the number of boxes of size ε needed to cover it is no more than $L(1/\varepsilon)$, where L is a constant depending on the length of the interval.

Similarly in dimension d, the number is $L(1/\varepsilon)^d$.

Now we can take different objects and ask how many boxes we need to cover them. If we denote by $N(\varepsilon)$ the number of boxes of side-length ε needed to cover a given set S, we want to say that S is d-dimensional when

$$N(\varepsilon) = L(1/\varepsilon)^d.$$

We notice, that d can be other than integer.

Box-counting dimension

Solving for d we obtain:

$$d = \frac{\ln N(\varepsilon) - \ln L}{\ln 1/\varepsilon}.$$

If L is a constant, we can omit it in the previous formula for small ε.

Definition. A bounded set S in $\mathbb{R}^m$ has *box-counting dimension*

$$\operatorname{boxdim}(S) = \lim_{\varepsilon \to 0} \frac{\ln N(\varepsilon)}{\ln 1/\varepsilon},$$

when the limit exists.

Example 1 Line

For a line segment in the plane of length l:

We need at least l/ε (lies horizontally or vertically) and at most $2l/\varepsilon$ (lies diagonally).

Then, $N(\varepsilon)$ is between $l(1/\varepsilon)$and $2l(1/\varepsilon)$.

The definition above gives $d = 1$.

Show that the box-counting dimension of a disk is 2.

Example 2 Cantor set

For the cantor set, at the step n the set C_n consists of 2^n intervals of length $1/3^n$.

It contains the endpoints of all 2^n intervals and they lie 3^{-n} far from each other.

$$\operatorname{boxdim}(C) = \lim_{\varepsilon \to 0} \frac{\ln 2^n}{\ln 3^n} = \frac{\ln 2}{\ln 3}.$$

It should be not surprising that the same result is obtained for the Sierpinski gasket.

Last slide!

Simplifications to the definition:

We can have other boxes than squares, other simplifications that allow easier calculation.

It is also sufficient to check $\varepsilon = b_n$ where $\lim_{n\to\infty} b_n = 0$ and $\lim_{n\to\infty} \frac{\ln b_{n+1}}{\ln b_n} = 1$.

Other definitions of fractal dimensions (correlation dimension) that allow easier calculation in some cases.