EVROPSKÁ UNIE
MINISTERSTVO ŠKOLSTVÍ, MLÁDEŽE A TĚLOVÝCHOVY
OP Vzdělávání pro konkurenceschopnost
VYSOKÉ UČENÍ TECHNICKÉ V BRNĚ

INVESTICE DO ROZVOJE VZDĚLÁVÁNÍ

Stochastic Model Predictive Control for Buildings

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20. 4. 2012

Tato prezentace je spolufinancována Evropským sociálním fondem a státním rozpočtem České republiky.

Motivation

Motivation

- Some aspects have been omitted in the last presentation..
  - Disturbances acting on the system are usually of stochastic nature
    - Weather forecast
    - Occupancy
  - The ISO norm specifying thermal conditions to be satisfied in the buildings says that the temperature range must be fulfilled in 95% of time instants.

Probability of comfort range violation

- This can be formulated in a stochastic programming framework as

$$P(f(x,w) \leq 0) > 1 - \alpha$$

  - P is a cumulative distribution function
  - $f(x,w) \leq 0$ is a constraint to be fulfilled
  - α is a tuning parameter
- But until now, we have had only deterministic model of the system
- w is the missing term..
- Deterministic model is extended by a stochastic part

$$x_{k+1} = Ax_k + Bu_k + Vv_k + w_k$$

- Assume that we have measurements and historical weather forecast
- Using regression analysis, the disturbance model can be obtained
- The actual disturbance acting on the system is decomposed as

$$v_k = \overline{v}_k + \tilde{v}_k$$

- The error of the weather forecast $\tilde{v}_k$ is colored noise, i.e. it is a result of the filtration

$$\tilde{v}_{k+1} = F\tilde{v}_k + Kw_k$$

where w_k follows Gaussian distribution with $\mathcal{N}(0, I)$

Description of the additive noise: weather

$$\tilde{v}_{k+1} = F\tilde{v}_k + Kw_k$$

- Parameters F and K are to be identified using linear regression
- Stochastic model has following form:

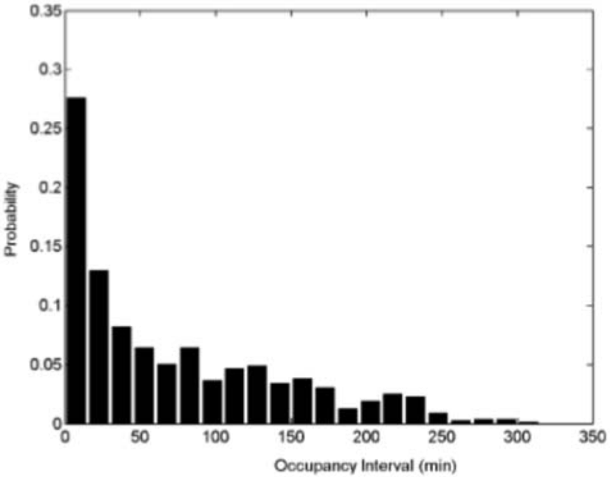
$$\begin{bmatrix} x_{k+1} \\ \tilde{v}_{k+1} \end{bmatrix} = \begin{bmatrix} A & V \\ 0 & F \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u_k + \begin{bmatrix} V \\ 0 \end{bmatrix} \overline{v}_k + \begin{bmatrix} 0 \\ F \end{bmatrix} w_k$$

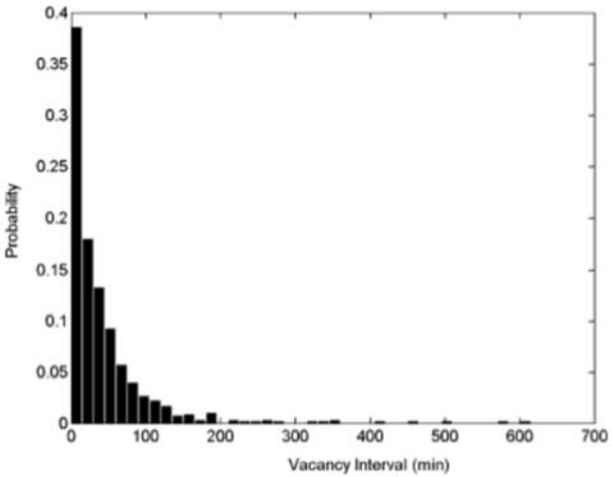
- Hereafter, the model will be written as

$$x_{k+1} = Ax_k + Bu_k + Vv_k + Hw_k$$

- Statistical nature of occupancy is usually neglected, but
- it can be handled in a similar way
- Occupancy and vacancy intervals follow the Poisson distribution

$$f(y) = \frac{1}{\beta} e^{\frac{-y}{\beta}}$$





Modifications to the OCP

Small modifications have to be done to our optimal control problem

$$\min \mathbf{E} \left\{ \sum_{k=0}^{N_u-1} |R_k u_k|_1 \right\}$$

subject to:

$$
\begin{aligned}
G_k u_k &\leq h_k \\
x_{k+1} &= Ax_k + Bu_k + Vv_k + Hw_k \quad k = 1 \ldots N_y \\
y_k &= Cx_k + Du_k + Wv_k \quad k = 1 \ldots N_y \\
x_0 &= x_{init} \\
P(\underline{r}_k \leq y_k &\leq \overline{r}_k) \geq 1 - \alpha \quad k = 1 \ldots N_y
\end{aligned}
$$

Stochastic Optimal Control

Stochastic optimal control problem

$$J = \mathbf{E}\left\{ l_N(x_N) + \sum_{k=0}^{N-1} l_k(x_k, u_k) \right\}$$

$$x_{k+1} = f(x_k, u_k, w_k)$$

- $x_0, w_0, \ldots, w_{N-1}$ are random variables
- $u_k = \phi_k(x_0, w_0, \ldots, w_{k-1})$
- goal: minimize J over all admissible causal policies $\phi_0, \ldots, \phi_{N-1}$.

- $I_k := (x_0, w_0, \dots, w_{k-1}, x_k)$
- $V_N(I_N) := l_N(x_N)$

$$V_k(I_k) = \min_{u_k} \{ l_k(x_k, u_k) + \mathbf{E}[V_{k+1}(I_k, w_k, f(x_k, u_k, w_k)) | I_k] \}$$

$$\phi_k^*(I_k) = \arg\min_{u_k} \{ l_k(x_k, u_k) + \mathbf{E}[V_{k+1}(I_k, w_k, f(x_k, u_k, w_k)) | I_k] \}$$

- with w_k and x_0 independent, optimal state feedback $\phi_t^*(I_k) = \varphi_k^*(x_k)$ exists
- as a computational tool tractable for very small problems only

Certainty equivalent control

- $w_0 \dots w_{N-1}$ replaced with estimates $\hat{w}_0 \dots \hat{w}_{N-1}$
- optimal for linear dynamics and quadratic cost functions.

- suboptimal for constrained quadratic control even in a receding horizon mode
- widely, and often unwittingly, used because of simplicity

Disturbance feedback policies

- finite basis $\mathcal{B}_k = (e_k^1, \ldots, e_k^{|\mathcal{B}_k|})$ chosen at each time step to construct a policy

$$\phi_k(w_0, \ldots, w_{k-1}) = \sum_{k=0}^{|\mathcal{B}_k|} \alpha_k^i e_k^i(w_0, \ldots, w_{k-1})$$

- with the coefficients α_k^i as optimization variables
- convex if stage cost is convex and dynamics linear
- tractable representation exists for quadratic cost and linear dynamics
- for affine-like policies approx. $mnN^2/2$ variables (but limited recourse can be used)
- for quadratic-like policies $\sim mn^2N^3$ variables
- input constraints and unbounded disturbances $\rightarrow e_k^i$ bounded

Chance constraints

- $P(f(x,w) \leq 0) > 1 - \alpha$
  - generally nonconvex
- individual: $P(a_i^T w \leq b_i) > 1 - \alpha_i$
  - exact second order cone / affine representation for affine DF / OL policies if w is Gaussian with independent components and $\alpha_i \leq 0.5$
  - robust approximation – avoids solving an SOCP but may be too conservative

Chance constraints

- $P(f(x,w) \leq 0) > 1-\alpha$
  - generally nonconvex
- joint: $P(Aw \leq b) > 1-\alpha$
  - no exact tractable representation (problematic constraint evaluation)
  - ellipsoidal approximation – usually very conservative
  - approximation by individual constraints $\rightarrow$ risk allocation

$$P\left(\bigcup_i P(a_i^T w > b_i)\right) \leq \sum_i P(a_i^T w > b_i) = \sum_i \alpha_i \leq \alpha$$

  - Can be optimized online in the OL setting (normal cdf among constraints though)
  - Can be too aggressive in receding horizon mode

1-norm stochastic optimal control

1-norm stochastic control problem

$$
\begin{aligned}
\underset{\eta,K}{\text{minimize}} \quad & \mathbf{E}\left\{ ||Q_N x_N||_1 + \sum_{k=0}^{N-1} ||Q_k x_k||_1 + ||R_k u_k||_1 \right\} \\
\text{subject to} \quad & u = \eta + K e(w) \\
& x_{k+1} = A x_k + B u_k + w_k \\
& K \text{ is strictly block lower triangular} \\
& |\eta_i| + \varepsilon ||K_i||_\infty \leq U_{\max}, \; i = 1, \ldots, mN
\end{aligned}
$$

- $e : \mathbb{R}^{nN} \to \mathbb{R}^{nN}$, $||e(w)||_\infty \leq \varepsilon$
- stochastic optimization techniques needed to solve “exactly” (e.g. stochastic subgradient) – slow convergence, but can be used for arbitrary disturbance distribution

1-norm stochastic control problem approximation

- $\eta + Ke(w)$ replaced with $\eta + Kw$ when computing K, η
- same constraints on η, K $\rightarrow$ input constraints will be satisfied when the original policy $u = \eta + Ke(w)$ is used
- analytic expression for the cost can be derived if the disturbances are jointly Gaussian
- state and control are affine in w $\rightarrow$ cost can be written as $\sum_i \mathbf{E}|\mu_i(\eta, K) + c_i^T(\eta, K)\tilde{w}|$ for $\tilde{w} \sim \mathcal{N}(0, I)$ with μ_i and c_i affine in η, K
- for $X \sim \mathcal{N}(\mu, \sigma)$ we have

$$\mathbf{E}|X| = \frac{1}{\sqrt{2\pi}} \left(2\,\sigma\, e^{-\frac{\mu^2}{2\sigma^2}} + \mu\sqrt{2\pi}\ \mathrm{erf}\left(\frac{\mu}{\sigma\sqrt{2}}\right)\right)$$

- $\mu(\eta, K) = \mu_i(\eta, K)$, $\sigma(\eta, K) = ||c_i(\eta, K)||_2$
- smooth in μ for $\sigma > 0$

Hessian

- for $X \sim \mathcal{N}(\mu(\eta, K), \sigma(\eta, K))$ with $\mu = \mu_0 + b^T \eta$, $\sigma = ||a + Ck||_2$, the Hessian of $(\mathbf{E}|X|)(\eta, K)$ is

$$\mathrm{Hess}(f) = \sqrt{\frac{2}{\pi}} e^{-\frac{\mu^2}{2\sigma^2}} \left(\frac{1}{\sigma} \begin{bmatrix} b \\ -q\frac{\mu}{\sigma} \end{bmatrix} \begin{bmatrix} b \\ -q\frac{\mu}{\sigma} \end{bmatrix}^T + \mathrm{Jac}(\nabla \sigma) \right),$$

- where

$$\mathrm{Jac}(\nabla\sigma) = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{||\sigma||_2} C^T \left(I - \frac{(a+Ck)(a+Ck)^T}{||\sigma||_2^2} \right) C \end{bmatrix},$$

$$q = \begin{bmatrix} 0 \\ C^T \frac{a+Ck}{\sigma} \end{bmatrix}$$

1-norm stochastic control problem approximation

- assume $w \sim \mathcal{N}(0, FF^T)$
- for $||Q_k x_k||_1$ terms we have

$$\sigma(q_{jk}^T x_k) = ||q_{jk}^T(\mathcal{C}_k + \mathcal{B}_k K F)||_2 = ||\mathcal{C}_k^T q_{jk} + (F^T \otimes q_{jk}^T \mathcal{B}_k) S k||_2,$$

$$\mu(q_{jk}^T x_k) = q_{jk}^T A^k x_0 + q_{jk}^T \mathcal{B}_k \eta,$$

- where

$$\mathcal{B}_k = [A^{k-1}B, \dots, B, 0, \dots, 0], \quad \mathcal{C}_k = [A^{k-1}, \dots, I, 0, \dots, 0]F,$$

- q_{jk} is j-th row of Q_k and S is a matrix of zeros and ones

- for $||R_k u_k||_1$ we have

$$\mu(r_{jk}^T u_k) = r_{jk}^T v_k \eta, \quad \sigma(r_{jk}^T u_k) = ||(F^T \otimes r_{jk}^T v_k) Sk||_2$$

- r_{jk} is j-th row of R_k and v_k selects k-th block row of η or K

Bound on suboptimality

- given (η, K), the difference between costs under the policies $u^e = \eta + Ke(w)$ and $u^w = \eta + Kw$ can be bounded
- for $||Qx_k||_1$ terms

$$\begin{aligned}
|\mathbf{E}(|q_j^T x_k^e| - |q_j^T x_k^w|)| &\leq \mathbf{E}\left||q_j^T x_k^e| - |q_j^T x_k^w|\right| \\
&\leq \mathbf{E}(|q_j^T x_k^e - q_j^T x_k^w|) = \mathbf{E}|q_j^T \mathcal{B}_k K(e(w) - w)| \\
&\leq ||q_j^T \mathcal{B}_k||_\infty ||K||_\infty \mathbf{E}||e(w) - w||_\infty \\
&\leq ||Q||_\infty ||\mathcal{B}_N||_\infty ||K||_\infty \mathbf{E}||e(w) - w||_\infty
\end{aligned}$$

Bound on suboptimality

- given (η, K), the difference between costs under the policies $u^e = \eta + Ke(w)$ and $u^w = \eta + Kw$ can be bounded
- similarly for $||Ru_k||_1$ terms

$$|\mathbf{E}(|r_j^T u_k^e| - |r_j^T u_k^w|)| \leq ||R||_\infty ||K||_\infty \mathbf{E}||e(w) - w||_\infty$$

- summing up all terms yields

$$|J_e - J_w| \leq (n_q(N+1)||Q||_\infty ||\mathcal{B}_N||_\infty + n_r N ||R||_\infty)\mathbf{E}||e(w) - w||_\infty$$

- where n_q and n_r are the numbers of rows of Q and R

Bound on suboptimality

- since $||K||_\infty \leq U_{\max}/\varepsilon$ for any feasible K, a suboptimality bound follows

$$J - J^* \leq 2(n_q(N+1)||Q||_\infty||\mathcal{B}_N||_\infty + n_r N||R||_\infty)\mathbf{E}||e(w) - w||_\infty \frac{U_{\max}}{\varepsilon}$$

- can be improved by terminating earlier in the string of inequalities
- $\mathbf{E}||e(w) - w||_\infty$ can be evaluated by Monte Carlo
- the bound will be small if $||e(w) - w||_\infty$ is large with only low probability
- viable choice for $e(w)$ – componentwise saturation

Computational issues

- can be solved by smooth interior-point methods with some additional care
- the problem is nondifferentiable for $\sigma = 0$, which can happen if
  - part of the state is not affected by the disturbance
  - R_k too large or control authority too small compared to $||e(w)||_\infty \to$ zero variance of k-th input
- remedy
  - replace the $||\cdot||_2$ in the expressions for σ by a smoothed approximation, e.g. $||x||_2 \approx \sqrt{1/n + \sum_i x_i^2}$
  - fix troublesome inputs during optimization when arrived at nondifferentiability and reoptimize with warm start

Example 1

- finite optimization horizon $T = 12$

$$A = \begin{bmatrix} 1 & -0.4 \\ 0.1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}, \quad \Sigma = I \otimes \begin{bmatrix} 8 & 5 \\ 5 & 6 \end{bmatrix}$$

- $Q = I,\ R = 0.1I$
- $\varepsilon = ||e(w)||_\infty = 4r = 13.92,\ r = \sqrt{\rho(\Sigma)} = 3.48$
- comparison of various control policies

Policy	DF (SH)	DF	CE-MPC (SH)	CE-MPC	$u = \eta$	CE-OL
J	86.8	92.1	98.3	119.2	140.4	143.9

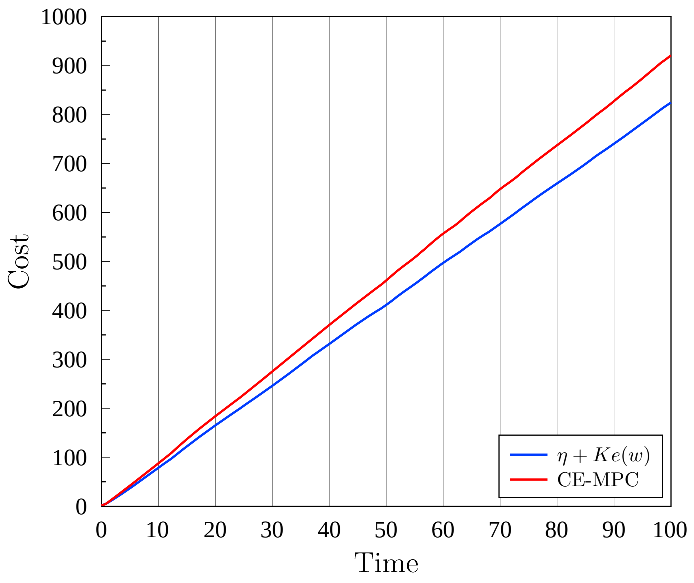
- DF – the disturbance feedback policy $u = \eta + Ke(w)$
- the bound for different values of ε

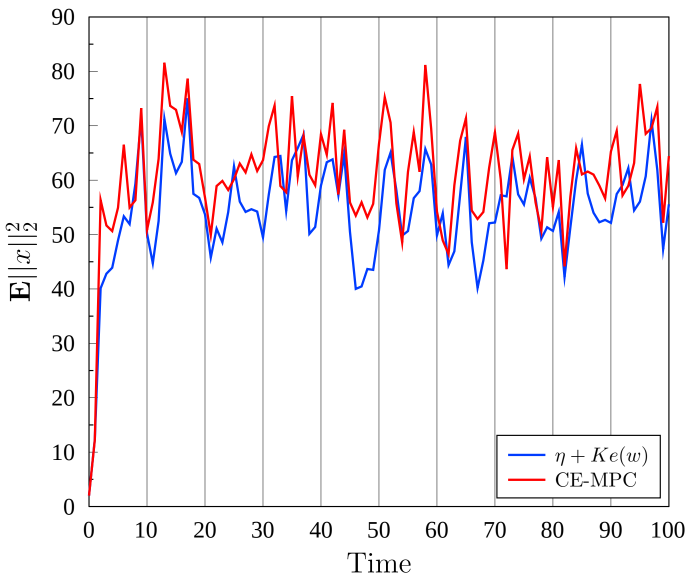
ε	$4r$	$3r$	$2.5r$	$2r$	$1r$
Bound	$4.69 \cdot 10^{-4}$	0.187	2.51	24.1	677.2
J_w^*	93.06	90.22	88.94	87.8	86.6
J_e	93.02	90.23	88.95	87.9	92.0

Example 2

- receding horizon mode with $N = 12$, $N_c = 1$ for CE-MPC and $N_c = 2$ for the disturbance feedback policy

$$A = \begin{bmatrix} 1 & 1 \\ -0.5 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{E}\{w_i w_j^T\} = \begin{bmatrix} 8 & 5 \\ 5 & 6 \end{bmatrix} \delta_{ij}$$





Conclusions

Conclusions

- The proposed algorithm not yet implemented on the real building – model of the disturbances is missing
- It is a convex and not so conservative as the other algorithms reported in literature
- possible extension to the general p-norm – an expression for $\mathbf{E}|X|^p$ needed

References